CYCLICALLY PRESENTED GROUPS AND RESULTANTS

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Abstract. We consider the family of irreducible cyclically presented groups on n generators whose generating word (in the standard rewrite) has length at most 15. Using the software packages KBMAG, **quotpic** and Magma, together with group and number theoretic methods, we show that if $6 \leq n \leq 100$ then the group is non-trivial. In an appendix we list the 47 cases within $2 \leq n \leq 100$ for which we know the group to be trivial, and 27 further cases for which triviality has yet to be determined.

1. Introduction

Let F_n denote the free group of rank n generated by $x_0, x_1, \ldots, x_{n-1}$ and let θ be the automorphism of F_n sending x_i to x_{i+1} (where subscripts are taken modulo n). Let w be a cyclically reduced element of F_n and define the group $G_n(w) = F_n/N$ where N is the normal closure in F_n of $\{w, w\theta, \ldots, w\theta^{n-1}\}$. A group G is said to be *cyclically presented* if $G \cong G_n(w)$ for some n and for some w.

The presentation for $G_n(w)$ is defined to be *irreducible* either when $n = 1$ or when $n \geq 2$ and the following two conditions are satisfied:

(1) w involves at least two of the x_i; and
(2) if w involves only $x_i, \ldots, x_{i_k}$ where $i_j < i_{j+1}$ $(1 \leq j \leq k-1)$ and where $k \geq 2$ then $\gcd(i_2 - i_1, \ldots, i_k - i_{k-1}, n) = 1$.

This idea was introduced in [6] to avoid the situation where the set of generators can be partitioned allowing the group to decompose into a free product of copies of some $G_m(w')$ for some word w', where m divides n.

In this paper we will be concerned mainly with extending the results of [7]. We will apply recent embedding theorems obtained in [9] and, in the spirit of [16] and [18], we will also apply recent number theoretic results [4] to the following problem studied in [6], [7], [8] and [10].

Problem 1. *For what values of n and for what w is $G_n(w)$ irreducible and trivial, that is, of order 1?*

This problem was inspired by a conjecture of Dunwoody [5] counter-examples to which were given in [6]. It is also worth mentioning that any $G_n(w)$ found to be trivial represent potential counter-examples to the Andrews–Curtis conjecture (for a discussion of which see for example [2]).

Two of the earliest examples (as far as we know) of irreducible trivial $G_n(w)$ for $n > 1$ are $G_n(x_0[x_0^{-1}, x_1])$ for $2 \leq n \leq 3$ (where $[a, b]$ denotes $a^{-1}b^{-1}ab$) discovered by Higman and Hirsch [11]. Further examples were exhibited in [6] and these included $G_4(x_0[x_1, x_3])$. This last example was generalised by Havas and Robertson [10] where in particular it was shown that, for all $k \geq 2$ and $m \geq 1$,

$$G_{km}\big(x_1^{-1}(x_0 x_m x_{2m} \ldots x_{(k-1)m})^{-1} x_m x_{2m} \ldots x_{(k-1)m} x_0\big)$$

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is irreducible and trivial. It follows that $G_n(x_0[x_i^\alpha, x_{i+k}^\alpha])$ is irreducible and trivial for $n = 2k \geq 4$, $\alpha \in \mathbb{Z} \setminus \{0\}$ and for each i such that $\gcd(i, k) = 1$ (see [8] for further discussion of these groups).

Now the automorphism θ of F_n defined above induces an automorphism of $G_n(w)$, and the resulting split extension $H_n(w)$, of $G_n(w)$ by the cyclic group of order n, has a presentation $H_n(w) = \langle x, t \mid t^n, \mathrm{w}(x,t)\rangle$ where $\mathrm{w}(x,t)$ is in the normal closure (in the free group on x and t) of x and t^n (see, for example, [15]). Conversely, it is readily verified that any group with such a presentation is a split extension of a group $G_n(w)$ for some w; in this case we say that $G_n(w)$ or w is the cyclically presented group or word *associated* with the word $\mathrm{w}(x,t)$.

Denote by $l = l\big(\mathrm{w}(x,t)\big)$ the *length* of $\mathrm{w}(x,t)$ regarded as a cyclically reduced word in the free group on x and t. In [7] it was shown that if $l \leq 15$, $6 \leq n \leq 10$ and the associated $G_n(w)$ is irreducible, then $G_n(w)$ is non-trivial. Our aim here is to relax the condition on n and consider the following special case of Problem 1.

Problem 2. *If $l \leq 15$ then for what n and w is $G_n(w)$ associated with $\mathrm{w}(x,t)$ both irreducible and trivial?*

Apart from a list of 27 open cases listed in Appendix B we will give a complete answer to Problem 2 for $2 \leq n \leq 100$. In fact we have found, for $l \leq 15$, a total of 47 words w such that $G_n(w)$ is both irreducible and trivial; these are listed in Appendix A. Our results have led us to expect that if $l \leq 16$, $6 \leq n$ and $G_n(w)$ is irreducible, then $G_n(w)$ is non-trivial. The following corollary to the results of Section 3 confirms this for $l \leq 15$ and $6 \leq n \leq 100$. (In fact for $l = 16$ there is a total of 125 words $\mathrm{w}(x,t)$ to be considered for $n \geq 6$. We have confirmed that all of these have non-trivial associated $G_n(w)$ except for one possible exception that the second-named author hopes to show infinite by geometric methods.)

Theorem 1.1. *If $G_n(w)$ is an irreducible cyclically presented group whose corresponding word $\mathrm{w}(x,t)$ has length at most 15, and if $6 \leq n \leq 100$, then $G_n(w)$ is non-trivial.*

We note that length 16 would be best possible, at least for $n = 6$, since if $\mathrm{w}(x,t) = xt^{-1}x^{-1}t^{-3}x^{-1}t^3xt^{-3}xt^{-2}$, a word of length 17, and $n = 6$, then the associated group is $G_6(x_0[x_1, x_4])$, which is one of the irreducible trivial examples of Havas and Robertson. In fact, if $\mathrm{w}(x,t)$ is the word associated with irreducible trivial $G_{km}(w)$ defined earlier, then $l\big(\mathrm{w}(x,t)\big) = 2k(m+1)+1$ for $k = 2$ and $l\big(\mathrm{w}(x,t)\big) = 2k(m+1)+3$ for $k > 2$ except when $(k,m) = (3,1)$ where $l\big(\mathrm{w}(x,t)\big) = 14$.

Our initial approach to Problem 2 is to repeat the experiment of [6], first without and then with what we call the determinant test; the details are given in Section 2. We use a number-theoretic approach (developed by the first-named author in [4] and recalled in Subsection 2.4) to determine, for a given word $\mathrm{w}(x,t)$, the values of n for which the determinant test passes. It is hoped that the method of subsection 2.4 will be of independent interest to the reader. In Section 3 we return to group theory and address Problem 2 for $n \leq 100$.

Remark. Throughout this paper we will be using the phrase 'we have confirmed' to mean we have used either previously known results from [6], [7], [8], [10] and [17] or we have used one of the software systems **quotpic** [12], KBMAG [13] or MAGMA [1]. Usually we first use KBMAG to try to confirm that G is an automatic group; if this is the case, KBMAG will then produce the order of G. If this is not successful, we use **quotpic** to search for subgroups of finite index, and use MAGMA to search for simple quotients. If such subgroups are found these two latter approaches will show G is non-trivial and in many of these cases we have confirmed G to be infinite by abelianizing the subgroup and checking the torsion-free rank.

For the 27 open cases listed in Appendix B, we have verified that none of the $H_n(w)$ or associated $G_n(w)$ has a confluent rewriting system, after running `kbprog` in KBMAG for at least 300000 equations—indeed, in all cases apart from $H_n(w)$ for (O22) and (O23), and $G_n(w)$ for (O8),(O23) and (O26), the number of word-differences exceeded 3000. We re-ran `kbprog` for each of these five cases for one million equations, and then implemented `gpmakefsa` in KBMAG, but with no success; each time the number of states exceeding 250000. We have also verified that none of the $H_n(w)$ has a subgroup of index k where $n < k \leq 20$, and none have a simple quotient of order up to 10^9 and degree at most 800.

2. The experiment, periodicity and number-theoretic results

2.1. **The experiment.** We repeated the experiment described in [6] with the parameters n slightly extended to $2 \leq n \leq 25$ and again with $l \leq 15$. For this purpose we have written a new Magma program which, for given l and n generates the set of words $\mathrm{w}(x,t)$ satisfying (R1)–(R7) below, and the subset which also passes the determinant test (R8).

Recall that in principle we check all words $\mathrm{w}(x,t)$ such that $l\big(\mathrm{w}(x,t)\big) \leq 15$ but can, without any loss, apply the following restrictions. Some of these ((R1), (R3), (R4)) are independent of n while the rest depend on n in a way which we will determine precisely.

- (R1) The word $\mathrm{w}(x,t)$ is cyclically reduced.
- (R2) The exponent sum of t in $\mathrm{w}(x,t)$ is congruent to 0 modulo n.
- (R3) The words $\mathrm{w}(x,t)$ are considered up to *equivalence* where $\mathrm{w}_1(x,t)$ is equivalent to $\mathrm{w}_2(x,t)$ if and only if $\mathrm{w}_1(x,t)$ can be obtained from $\mathrm{w}_2(x,t)$ by a sequence of the following moves:
  - (a) cyclic permutation;
  - (b) replace x by x^{-1} everywhere;
  - (c) replace t by t^{-1} everywhere;
  - (d) inversion.
- (R4) The exponent sum of x in $\mathrm{w}(x,t)$ is[1] equal to ± 1. [Otherwise, $G_n(w)$ is non-trivial.]
- (R5) No cyclic permutation of $\mathrm{w}(x,t)$ contains the subwords t^{-k} or t^{k+1} (if $n = 2k$), or $t^{-(k+1)}$ or t^{k+1} (if $n = 2k+1$).
- (R6) The associated $G_n(w)$ is irreducible.
- (R7) If $n \geq 4$ then the associated word w involves at least three of the x_i [19].
- (R8) The determinant of the relation matrix of the cyclic presentation for the associated $G_n(w)$ equals ± 1. [Otherwise, the abelianization of $G_n(w)$ is non-trivial.]

When enumerating all words satisfying these conditions, we can use (R3) to restrict to those which start with a positive power of x and end with a positive power of t, by applying suitable equivalence transformations. Words in this form are products of subwords of the form $x^i t^j$ where $i\,j \neq 0$; we call the exponents j which appear the *t-exponents* of the word. Now, condition (R2) is that the t-exponents of $\mathrm{w}(x,t)$ sum to 0 (mod n); condition (R5) is that each t-exponent j satisfies $-\frac{n}{2} < j \leq \frac{n}{2}$, and condition (R6) that the gcd of the t-exponents is coprime to n. Finally, condition (R7) excludes words whose sequence of t-exponents has the form $k, -k, k, -k, \ldots$ for some k.

[1] The condition stated in [6] was that the exponent sum be 1, which is not invariant under equivalence.

Let $W(l,n), S(l,n)$ (respectively) denote the set of words $\mathrm{w}(x,t)$ satisfying (R1)–(R7), (R1)–(R8) (respectively) for values l and n. The remarks above will allow us to determine when a word $\mathrm{w}(x,t) \in W(l,n)$ remains in $W(l,n')$ for $n' > n$.

The results of the experiment are given in Tables 1 and 2. The tables give the number of inequivalent words $\mathrm{w}(x,t)$ that remain after applying (R1) to (R8), and so words that may give rise to irreducible trivial $G_n(w)$:

- Table 1 shows $|W(l,n)|$, the number of words satisfying (R1)–(R7); this is 0 for $l < 5$.
- Table 2 shows $|S(l,n)|$, the number of words satisfying (R1)–(R7), together with the determinant test (R8); this is 0 for $l < 7$.

TABLE 1. $|W(l,n)|$ for $5 \le l \le 15$ and $2 \le n \le 25$.

$l \setminus n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
5	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
8	0	2	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	4	10	14	11	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14
10	0	9	0	15	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	9	25	66	72	109	89	94	94	92	94	92	94	92	94	92	94	92	94	92	94	92	94	92	94
12	0	53	0	85	0	60	0	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	26	139	420	427	680	580	688	624	621	630	619	630	619	630	619	630	619	630	619	630	619	630	619	630
14	0	227	0	678	0	438	0	164	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	82	657	2451	2739	4868	3917	4910	4263	4518	4349	4333	4357	4330	4355	4330	4357	4328	4357	4330	4355	4330	4357	4328	4357

TABLE 2. $|S(l,n)|$ for $7 \le l \le 15$ and $2 \le n \le 25$.

$l \setminus n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
7	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
8	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	1	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
10	0	3	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	2	2	4	9	4	6	4	4	4	7	4	7	4	4	4	7	4	7	4	4	4	7	4	7
12	0	4	0	7	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	7	22	17	36	21	42	21	24	23	41	21	38	23	21	21	38	21	38	21	24	23	38	21	35
14	0	21	0	34	0	17	0	2	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	20	63	103	206	105	208	111	118	120	198	109	190	120	109	109	190	109	190	109	118	120	190	109	181

Two things are clear from the tables: first, in both tables there is a significant difference in behaviour according to l being even or odd. Secondly, we will see in subsections 2.2 and 2.3 that for each fixed l, the behaviour as n increases is regular. In fact we will see that both $W(l,n)$ and $S(l,n)$ for $n > l-2$ are all determined by $W(l,l-2)$ and $S(l,l-2)$, and are in fact periodic with respect to n.

2.2. **Periodicity.** For fixed l, some of the conditions (R1)–(R8) are independent of n while some depend on n. We now examine this issue in more detail.

Lemma 2.1. *Let* $\mathrm{w} = \mathrm{w}(x,t) \in W(l,n)$. *Then*

(i) *The number of occurrences of* x *and* x^{-1} *in* w *is at least* 3 *(when* $l \ge 3$*).*

(ii) *If* $n \ge l-2$*, then the exponent sum of* t *in* $\mathrm{w}(x,t)$ *is* 0. *Hence if* l *is even, then* $W(l,n)$ *is empty, while if* l *is odd then condition (R2) holds for all* $n \ge l-2$ *once it holds for one such value.*

(iii) *If* $n \ge l-2$*, then for each subword* $x^i t^j$ *in* w *we have* $|j| \le (l-3)/2$. *Hence condition (R5) holds for all* $n \ge l-2$ *once it holds for one such value.*

Proof. (i) By (R4) the number is odd; if it is 1 then $\mathrm{w} = xt^{l-1}$, which cannot satisfy both $(R2)$ and (R5) for $l > 1$.

(ii) Hence the total number of occurrences of t and t^{-1} is at most $l-3 \leq n-1$, so the total t-exponent is less than n in absolute value; by (R2) it must be 0. When l is even, the total t-exponent is odd, giving a contradiction.

(iii) Let the t-exponents in the subwords be j_k for $1 \leq k \leq K$. By the previous part the sum s of the positive j_k is minus the sum of the negative j_k. The number of occurrences of t and t^{-1} is then $2s \leq l-3$; so for each k we have $|j_k| \leq s \leq (l-3)/2$.

□

The case of even l may now be dealt with.

Proposition 2.2. (i) *If l is even, $l \geq 8$ and $n \geq l-2$, then $|S(l,n)| = 0$.*

(ii) *If l is even, $8 \leq l \leq 14$ and $n \geq 2$, then the total number of words* $\mathrm{w}(x,t)$ *that need to be considered is* 97.

Proof. (i) follows from Lemma 2.1 (ii). Then for (ii), we need only sum the appropriate entries in Table 2. □

We consider now the case when l is odd. Unlike the even case it will turn out (subsection 2.3) for each fixed l that $|S(l,n)| > 0$ for infinitely many values of n. However, both $W(l,n)$ and $S(l,n)$ for all $n \geq l-2$ are determined by $W(l,l-2)$ and $S(l,l-2)$.

First we consider $W(l,n)$, ignoring condition (R8). For l odd, $l \geq 7$ and $n \geq l-2$, Lemma 2.1 implies that both (R2) and (R5) are now independent of n, so for all $n \geq l-2$ the situation essentially stabilises.

Lemma 2.3. *Let $l \geq 7$ be odd, and let* $\mathrm{w} = \mathrm{w}(x,t) \in W(l,n_0)$ *for some $n_0 \geq l-2$. Then*

(i) w *satisfies (R2) and (R5) for all $n \geq l-2$;*

(ii) *let g be the* gcd *of the t-exponents in* $\mathrm{w}(x,t)$*. If $g = 1$ then* $\mathrm{w}(x,t)$ *satisfies (R6) for all $n \geq l-2$; otherwise,* $\mathrm{w}(x,t)$ *satisfies (R6) for n if and only if* $\gcd(g,n) = 1$.

(iii) w *satisfies (R7) for all $n \geq l-2$;*

Proof. (i) is immediate from Lemma 2.1, while (ii) and (iii) follow from the characterisations of (R6) and (R7) given above. □

Using this lemma, we see that for $n \geq l-2$ the sets $W(l,n)$ either stabilise or at worst become periodic in n (on account of (R6)). Precisely, we have the following.

Proposition 2.4. *Let $l \geq 7$ be odd. Then for all $n \geq l-2$, the sets $W(l,n)$ are periodic in n. For $l \leq 15$ we have $W(l,n) \subseteq W(l,l-2)$, and the periods are* $1, 1, 2, 2, 6$ *for* $l = 7, 9, 11, 13, 15$ *respectively.*

Proof. By the previous lemma, a word $\mathrm{w}(x,t) \in W(l,l-2)$ will remain in $W(l,n)$ for all $n > l-2$ provided that condition (R6) is satisfied. Looking at the sets $W(l,l-2)$ produced by our program, we find that for $l = 7$ and $l = 9$, all have the gcd of the t-exponents equal to 1, so that $W(l,n) = W(l,l-2)$ for all $n \geq l-2$ for these l.

For $l = 11$, we find that of the 94 words in $W(11,9)$, 92 have gcd $= 1$, while two have gcd $= 2$; it will also be relevant later to observe that no candidates for inclusion in $W(11,9)$ were excluded by (R6) (that is, no words with gcd $= 3$ are generated). Hence for all odd $n \geq 9$ we have $W(11,n) = W(11,9)$, of size 94, while for all even $n \geq 10$ we have $W(11,n) = W(11,10)$, of size 92.

For $l = 13$, there is a similar picture, where 619 words have gcd $= 1$ and a further 11 have a gcd $= 2$; hence the size of $W(13,n)$ alternates between 630 and 619 according to the parity of n.

For $l = 15$, there are 4357 words in all, including 27 with gcd = 2 and two with gcd = 3 (the remaining words having gcd = 1). Hence for $n \geq 13$ the sets $W(15, n)$ depend on $n \pmod 6$, with sizes as given in Table 1.

Finally, we must show that $W(l, n) \subseteq W(l, l-2)$ for all $n \geq l-2$. This will follow provided that no words are excluded from $W(l, l-2)$ by condition (R6) alone. This is obvious for $l \in \{7, 9, 13, 15\}$ since then $l - 2$ is prime. It also holds for $l = 11$, since as we observed above, it is impossible for the gcd of the t-exponents to be 3 in this case. □

Remark. For larger odd l, when $l-2$ is not prime we may not have $W(l, n) \subseteq W(l, l-2)$ for all $n \geq l-2$. For example, when $l = 17$, the same argument as above shows that $W(17, n) \subseteq W(17, 17)$ for all $n \geq 15$, but $W(17, 15)$ is strictly smaller on account of words failing (R6) with a gcd of 3, for example $x^3t^{-3}x^{-1}t^{-3}x^{-1}t^6$.

For $l = 5$, the single word in $W(5, 3)$ does not appear for $n \geq 4$ since it fails condition (R7).

Using these theoretical results together with the (finite) computation which went into compiling the entries where $n \leq l - 2$ in Table 1, we have now determined the sets $W(l, n)$ for infinitely many pairs (l, n), satisfying $l \leq 15$ and $n \geq 2$.

2.3. **The determinant test.** Now we turn our attention to $S(l, n)$. We will say that $\mathrm{w}(x, t) \in W(l, n)$ *fails the determinant test* if condition (R8) fails, *i.e.*, the determinant of the relation matrix of the associated $G_n(w)$ is *not* equal to ± 1. The words in $S(l, n)$ are precisely those in $W(l, n)$ which pass the determinant test.

We have shown that for $n \geq l-2$ the $W(l, n)$ are periodic in n. We now show that the determinant condition is also periodic for these words. To do this, we recall that the *polynomial associated with* $\mathrm{w}(x, t)$ is defined to be $f_{\mathrm{w}}(s) = \sum_{i=0}^{n-1} a_i s^i \in \mathbb{Z}[s]$ where a_i is the exponent sum of x_i in the word w associated with $\mathrm{w}(x, t)$ (that is, the word obtained when $\mathrm{w}(x, t)$ is written in terms of the x_i). Define

$$\mathcal{P}(l, n) = \{f_{\mathrm{w}}(s) \mid \mathrm{w} \in W(l, n)\}.$$

The fact that we do not work modulo n in obtaining $f_{\mathrm{w}}(s)$ from $\mathrm{w}(x, t)$ is significant, since it will allow us to deal with all n at once (at least for $n \geq l - 2$).

If all the roots of the polynomial $f_{\mathrm{w}}(s)$ are roots of unity the word w will be called *cyclotomic* (in fact in general $f_{\mathrm{w}}(s)$ will be either a cyclotomic polynomial or a product of these); otherwise w is called *non-cyclotomic*.

It turns out that there are two quite different behaviours depending on being cyclotomic and non-cyclotomic.

In the cyclotomic case we show in this subsection that there are infinitely many n for which w passes the determinant test, and thus lies in $S(l, n)$, forming a periodic sequence. We include as a degenerate cyclotomic case words with polynomial the constant $f_{\mathrm{w}}(s) = 1$, since this has no roots which are not roots of unity. In the cyclotomic case the periodicity of $S(l, n)$ depends on the order of the roots of unity involved, and we will see (Proposition 8) that the cyclotomic words in $S(l, 13)$ already include all those in $S(l, n)$ for all $n \geq l - 2$.

In the non-cyclotomic case, we will show in the next subsection that, although each non-cyclotomic word in $W(l, l - 2)$ recurs in $W(l, n)$ for infinitely many n, it can only be in $S(l, n)$ for finitely many n – such words we call *sporadic*. Moreover, for all odd l with $l \leq 15$, there are in fact no non-cyclotomic words which pass the determinant test for any $n \geq l - 2$, with the exception of these three words in $W(13, 11)$:

$$xtx^{-1}t^{-4}x^{-1}txtxt, \qquad xt^2x^{-1}t^{-1}xt^{-3}x^{-1}txt, \qquad xtx^{-1}t^{-2}xt^{-2}x^{-1}txt^2;$$

these all have polynomial $f(s) = s^4 - s^3 - s^2 - s + 1$, and we will show in the next section (using the methods of [4]) that they are in $S(13, 11)$ but not in $S(13, n)$ for any $n > 11$. Apart from these exceptions, all words in $S(l,n)$ for all $n \geq l-2$ are cyclotomic.

Taken together, this will leave a finite number of sporadic words in $S(l,n)$ for $n \leq l-3$ together with three in $S(13,11)$ and another finite number of cyclotomic words in $S(l,n)$ which are in $S(l,n)$ for infinitely many $n \geq l-2$. These finitely many words will be considered further in the final section of the paper.

The following theorem summarises these results. The proof will take up the rest of this subsection and the next.

Theorem 2.5. *For all odd $l \leq 15$ and all $n \geq 13$,*

$$S(l,n) = W(l,n) \cap S(l,13).$$

In particular, $S(l,n) \subseteq S(l,13)$ for all $n \geq 13$ and all $l \leq 15$.

We remark that this theorem also holds for all $n \geq l-2$ except for the pair $(l,n) = (13,11)$.

For future reference, the sporadic words occur as follows.

Lemma 2.6. *Let $l \leq 15$ be odd. Then:*

(i) $\mathrm{w}(x,t) \in S(l,13)$ *if and only if* $\mathrm{w}(x,t)$ *is not sporadic;*
(ii) *there is a total of* 209 *sporadic words of length* ≤ 15*, counting multiplicity, consisting of* 2, 7, 36, 164 *of length* 9, 11, 13, 15 *(respectively).*

Proof. (i) This follows from Theorem 2.5 and its proof in subsection 2.4.
(ii) Thus for odd $l \leq 15$, the sporadic words are precisely those $\mathrm{w}(x,t)$ in $S(l,n)$ for some $n \leq 12$ that are not in $S(l,13)$. Comparing the lists of computed words gives the result, noting for each sporadic word the values of n for which it occurs.

□

We now consider the cyclotomic words.

TABLE 3. Cyclotomic polynomials associated to non-sporadic words, with multiplicities

	$l \leq 7$	9	11	13	15
1	0	0	4	21	109
$\Phi_6(s) = 1 - s + s^2$	1	1	2	11	59
$\Phi_{12}(s) = 1 - s^2 + s^4$	0	0	1	3	12
$\Phi_{10}(s) = 1 - s + s^2 - s^3 + s^4$	0	0	0	3	9
$\Phi_{18}(s) = 1 - s^3 + s^6$	0	0	0	0	1

The list of $f_{\mathrm{w}}(s)$ for cyclotomic (non-sporadic) $\mathrm{w}(x,t) \in S(l,l-2)$ is given in Table 3, together with the multiplicity with which each one occurs (working up to equivalence of words). We denote by Φ_m the mth cyclotomic polynomial; as noted earlier, it is convenient to treat the constant polynomial 1 as cyclotomic in this context. These will yield determinant ± 1 for the following values of n (see, for example, [18], Theorem 1):

1	all $n \geq 2$;
$\Phi_6(s)$	$n \equiv \pm 1 \pmod 6$;
$\Phi_{12}(s)$	$n \equiv \pm 1, \pm 2, \pm 5 \pmod{12}$ (but see below);
$\Phi_{10}(s)$	$n \equiv \pm 1, \pm 3 \pmod{10}$;
$\Phi_{18}(s)$	$n \equiv \pm 1, \pm 3, \pm 5, \pm 7 \pmod{18}$ (but see below).

However, not all the words $w(x,t)$ with these polynomials are irreducible. Three words with Φ_{12} fail to be irreducible for even n, since their t-exponents are all even, namely $xt^{-4}xt^2x^{-1}t^2$ with $l=11$, $xt^{-2}xt^{-2}xt^2x^{-2}t^2$ with $l=13$, and $xt^{-2}x^2t^{-2}xt^2x^{-3}t^2$ with $l=15$. Hence these only contribute to $S(l,n)$ when $n\equiv\pm1,\pm5 \pmod{12}$; that is, when $n\equiv\pm1 \pmod 6$.

Similarly, the polynomial Φ_{18} only occurs for one word, namely $xt^{-6}xt^3x^{-1}t^3$, which is not irreducible for $n\equiv 0 \pmod 3$; it follows that this word only contributes to $S(l,n)$ when $n\equiv\pm1,\pm5,\pm7 \pmod{18}$, that is, when $n\equiv\pm1 \pmod 6$.

Putting all of this together we get the following result, where the numbers generally depend on $n \pmod{30}$, but there is an adjustment when $n\equiv\pm2 \pmod{12}$ and we include the three sporadic non-cyclotomic words in $S(13,11)$.

Proposition 2.7. *For $n\geq l-2$ we have:*

$$|S(7,n)| = \begin{cases} 1 & \text{if } n\equiv\pm1 \pmod 6 \\ 0 & \text{otherwise;} \end{cases}$$

$$|S(9,n)| = \begin{cases} 1 & \text{if } n\equiv\pm1 \pmod 6 \\ 0 & \text{otherwise;} \end{cases}$$

$$|S(11,n)| = \begin{cases} 7 & \text{if } n\equiv\pm1 \pmod 6 \\ 4 & \text{otherwise;} \end{cases}$$

$$|S(13,n)| = \begin{cases} 41 & \text{if } n=11 \\ 38 & \text{if } n\equiv\pm1,\pm7,\pm11,\pm13 \pmod{30} \text{ but } n\neq 11 \\ 24 & \text{if } n\equiv\pm3,\pm9 \pmod{30} \\ 35 & \text{if } n\equiv\pm5 \pmod{30} \\ 23 & \text{if } n\equiv\pm2 \pmod{12} \\ 21 & \text{otherwise;} \end{cases}$$

$$|S(15,n)| = \begin{cases} 190 & \text{if } n\equiv\pm1,\pm7,\pm11,\pm13 \pmod{30} \\ 118 & \text{if } n\equiv\pm3,\pm9 \pmod{30} \\ 181 & \text{if } n\equiv\pm5 \pmod{30} \\ 120 & \text{if } n\equiv\pm2 \pmod{12} \\ 109 & \text{otherwise.} \end{cases}$$

We postpone the group theoretic results for $\mathrm{w}(x,t)$ of length at most 15 and for $2\leq n\leq 100$ until the final section.

2.4. **Resultants and the proof of Theorem 2.5.** In this section we use results from [4] to finish the proof of Theorem 2.5.

Let $\mathrm{w}(x,t)\in W(l,k)$ for some $k\geq 2$ and let $f_{\mathrm{w}}(s)$ be the polynomial associated with $\mathrm{w}(x,t)$ as defined in subsection 2.3. If for $n\geq k$ we denote the relation matrix of the associated $G_n(\mathrm{w})$ by $M_n(f_{\mathrm{w}})$, then $\mathrm{w}(x,t)$ is said to fail the determinant test for the value n precisely when $\det(M_n(f_{\mathrm{w}}))\neq\pm1$.

Using the fact that $M_n(f_{\mathrm{w}})$ is a circulant matrix it is shown in [15, page 77] that $\det(M_n(f_{\mathrm{w}})) = \prod_{\zeta:\zeta^n=1} f_{\mathrm{w}}(\zeta)$. The key observation made in [4] is that using standard properties of the resultant, $\mathrm{Res}(g,h)$ of two polynomials $g(s)$ and $h(s)$ we have[2]

$$\prod_{\zeta:\zeta^n=1} f_{\mathrm{w}}(\zeta) = \mathrm{Res}(f, s^n - 1) = (-1)^{nd} a_d^n \prod_{\beta: f_{\mathrm{w}}(\beta)=0} (\beta^n - 1)$$

where the coefficient $a_d \neq 0$ but $a_i = 0$ for $i > d$. So, setting $B(f_{\mathrm{w}}, n) = |\mathrm{Res}(f, s^n - 1)|$, we have $B(f_{\mathrm{w}}, n) = \pm \det M_n(f_{\mathrm{w}})$, and we are interested in determining for what values of n, $B(f_{\mathrm{w}}, n) = 1$. A key observation of [4] is that there are only finitely many such n unless all the roots of f_{w} are roots of unity (including $f_{\mathrm{w}} = 1$); and when this is not the case a method was given in [4] for determining all the values of n for which $B(f_{\mathrm{w}}, n) = 1$.

It follows that the proof of Theorem 2.5 reduces to an analysis of $B(f_{\mathrm{w}}, n)$ for $f_{\mathrm{w}} \in \mathcal{P}(l, l-2)$ (associated with $w(x,t) \in W(l, l-2)$) where l is odd and $l \leq 15$. The cyclotomic cases have been covered in the previous section. Applying the method of [4] we can determine, for each non-cyclotomic polynomial f in $\mathcal{P}(l, l-2)$, the maximum value of n for which $B(f, n) = 1$. The results are given in Table 4.

TABLE 4. Numbers of cyclotomic and non-cyclotomic polynomials in $\mathcal{P}(l, l-2)$

l	#	# cyclo.	# non-cyclo.	max. n
7	2	1	1=1+0	2
9	11	1	10=10+0	5
11	54	3	51=50+1	7
13	209	4	205=203+2	11
15	735	5	730=723+7	10

Each row of Table 4 shows the total number of distinct polynomials in $\mathcal{P}$, the number which are cyclotomic (from Table 3) and the remaining number of non-cyclotomic polynomials. In the last column we give the maximum value of n for which $B(f, n) = 1$ for any $f \in \mathcal{P}(l, l-2)$. Note that this number is strictly less than $l - 2$ in each case, except when $l = 13$, and is always strictly less than 13.

The number of non-cyclotomic polynomials is expressed as a sum of two terms. The first summand shows the number of polynomials in the set which have no root on the unit circle, or which are reducible and have a factor with no such root. For such polynomials, determining the values of n for which $B(f, n) = 1$ is more elementary, using the following result from [4], together with the observation that if f factors over $\mathbb{Z}$ as $f = f_1 f_2$ then $B(f, n) = B(f_1, n) B(f_2, n)$.

Proposition 2.8. *Let $f \in \mathbb{Z}[x]$ have leading coefficient $a > 0$, no repeated roots and no roots on the unit circle. Assume that $f(0) \neq 0$. Set*

$$S = \{\beta \in \mathbb{C} \mid f(\beta) = 0,\ |\beta| < 1\},\ s = \#S,$$
$$R = \{\beta \in \mathbb{C} \mid f(\beta) = 0,\ |\beta| > 1\},\ r = \#R.$$

Let n_0 be an integer such that

$$n_0 \geq \begin{cases} \max\left(\dfrac{s \log 2}{\log a}, \max\limits_{\beta \in S}\left\{\dfrac{\log 2}{-\log \beta}\right\}\right) & \text{if } r = 0 \\ \max\left(\max\limits_{\beta \in S}\left\{\dfrac{\log 2}{-\log \beta}\right\}, \max\limits_{\beta \in R}\left\{\dfrac{\log c}{\log \beta}\right\}\right) & \text{if } r > 0 \end{cases}$$

where $c = 2^{s/r} + 1$. Then $B(f, n) \neq 1$ for all $n > n_0$.

[2] In the second product, each root β is taken with multiplicity, so the number of factors is exactly $\deg f_{\mathrm{w}}$.

In each case, we calculated n_0 from the roots of f_{w}, and then computed $B(f_{\mathrm{w}}, n)$ for $n \leq n_0$ using resultants, to find the complete set of n for which $B(f_{\mathrm{w}}, n) = 1$.

This leaves polynomials which are such that all their irreducible factors have at least one root on the unit circle but are not cyclotomic. The number of these for each l is shown as the second summand in the non-cyclotomic column of Table 4. There are seven such distinct polynomials for $l \leq 15$, which are all in fact irreducible:

$$
\begin{array}{ll}
p_1 = 3s^2 - 5s + 3 & p_2 = 2s^4 - 3s^2 + 2 \\
p_3 = 2s^4 - s^3 - s^2 - s + 2 & p_4 = 2s^2 - 3s + 2 \\
p_5 = s^4 + s^3 - 3s^2 + s + 1 & p_6 = s^4 - s^3 - s^2 - s + 1 \\
p_7 = s^4 - 2s^3 + s^2 - 2s + 1 &
\end{array}
$$

Each of these is in $\mathcal{P}(15, 13)$, while p_4 also occurs in $\mathcal{P}(11, 9)$ and in $\mathcal{P}(13, 11)$, and p_6 also in $\mathcal{P}(13, 11)$. These polynomials all appear in the table at the end of [4]. In each case, a p-adic method based on Strassmann's Theorem was used to show that a given finite set of n for which $B(f, n) = 1$ was complete. The prime p has to be chosen to satisfy certain technical conditions, notably that the polynomial f splits into distinct linear factors modulo p: see [4] for details. For example, using $p = 547$ one may show that $B(p_5, n) \neq 1$ except for $n = 1$.

We can read off from the table in [4] the maximum values of n for which $B(p_j, n) = 1$ for each j. The only case where this maximum value is not strictly less than $l - 2$, for a polynomial in $\mathcal{P}(l, l-2)$, is for $p_6 \in \mathcal{P}(13, 11)$ where the maximum is $n = 11$. There are three words in $W(13, 11)$ which have this polynomial, namely the ones listed in the previous section.

In this way we obtained the entries in the final column of Table 4, and complete the proof of Theorem 2.5.

3. Group-theoretic results

Let $l = l\big(\mathrm{w}(x, t)\big) \leq 15$ be even. By Proposition 2.2 (ii) there are 97 words $\mathrm{w}(x, t)$ to be considered. Of the 97 associated $G_n(w)$ there are 7 examples of order 1 and these are (T1)–(T7) of Appendix A.

We have confirmed that 87 of the remaining 90 $G_n(w)$ are non-trivial (4 of order 120, 78 of infinite order, 5 of order greater than 1 whose exact order we have not yet established). All of this leaves 3 open cases and these are (O1)–(O3) of Appendix B.

Assume from now on that $l \leq 15$ is odd. By Lemma 2.6 (ii) there is a total of 209 sporadic $\mathrm{w}(x, t)$ to be considered and of these there are 24 examples of order 1, namely T(8)–T(31) of Appendix A.

Of the remaining 185 sporadic examples 174 of the associated $G_n(w)$ have been confirmed non-trivial (3 of order 120, 149 of infinite order, 22 of order greater than 1 whose exact order we have not yet established). This leaves 11 open cases and these are (O4)–(O14) of Appendix B.

Finally we are left to consider non-sporadic $\mathrm{w}(x, t)$. It can be seen by summing the entries in Table 3 that there is a total of 237 such words. We partition this set into words of *A-type* and words of *U-type*. A word $\mathrm{w}(x, t)$ is said to be of A-type if we have been able to show that the associated $G_n(w)$ is an infinite automatic group for all n such that n is prime and $11 \leq n \leq 97$; otherwise $\mathrm{w}(x, t)$ is said to be of U-type. So far we have managed to confirm using KBMAG that 216 of the

237 $\mathrm{w}(x,t)$ are of A-type. The 21 words of U-type are the following.

(U1)	$xt^2xt^{-1}x^{-1}t^{-1}$	$(l=7)$ $(n\geq 5)$
(U2)	$xtxtxt^{-1}x^{-2}t^{-1}$	$(l=9)$ $(n\geq 5)$
(U3)	$x^2txt^{-1}x^{-3}t^{-1}xt$	$(l=11)$ $(n\geq 5)$
(U4)	$xt^4xt^{-2}x^{-1}t^{-2}$	$(l=11)$ $(n\geq 10)$
(U5)	$x^3txt^{-1}x^{-4}t^{-1}xt$	$(l=13)$ $(n\geq 5)$
(U6)	$xt^2xt^2xt^{-2}x^{-2}t^{-2}$	$(l=13)$ $(n\geq 5)$
(U7)	$xtxt^3xt^{-2}x^{-1}t^{-1}x^{-1}t^{-1}$	$(l=13)$ $(n\geq 7)$
(U8)	$xt^2xt^2xt^{-1}x^{-1}t^{-2}x^{-1}t^{-1}$	$(l=13)$ $(n\geq 7)$
(U9)	$xt^4xt^{-1}x^{-1}t^{-1}xt^{-1}x^{-1}t^{-1}$	$(l=13)$ $(n\geq 9)$
(U10)	$x^2txtxt^{-2}x^{-1}t^2x^{-1}t^{-1}x^{-1}t^{-1}$	$(l=15)$ $(n\geq 4)$
(U11)	$x^2txtx^{-1}t^{-2}x^{-1}t^2xt^{-1}x^{-1}t^{-1}$	$(l=15)$ $(n\geq 4)$
(U12)	$x^2txt^{-2}xtx^{-1}t^{-1}x^{-1}t^2x^{-1}t^{-1}$	$(l=15)$ $(n\geq 4)$
(U13)	$x^2txt^{-2}x^{-1}tx^{-1}t^{-1}xt^2x^{-1}t^{-1}$	$(l=15)$ $(n\geq 4)$
(U14)	$x^2t^2xt^{-1}xt^{-1}x^{-1}tx^{-1}tx^{-1}t^{-2}$	$(l=15)$ $(n\geq 4)$
(U15)	$x^2t^2xt^{-1}x^{-1}t^{-1}x^{-1}txtx^{-1}t^{-2}$	$(l=15)$ $(n\geq 4)$
(U16)	$x^4txt^{-1}x^{-5}t^{-1}xt$	$(l=15)$ $(n\geq 5)$
(U17)	$xtxt^2xtxt^{-1}x^{-1}t^{-1}x^{-1}t^{-1}x^{-1}t^{-1}$	$(l=15)$ $(n\geq 5)$
(U18)	$xtxtx^{-1}txtxt^{-1}x^{-1}t^{-2}x^{-1}t^{-1}$	$(l=15)$ $(n\geq 5)$
(U19)	$x^2t^2xt^{-2}x^{-3}t^{-2}xt^2$	$(l=15)$ $(n\geq 5)$
(U20)	$x^2t^2xt^{-1}x^{-1}t^{-1}x^{-1}t^{-1}x^{-1}t^{-1}xt^2$	$(l=15)$ $(n\geq 7)$
(U21)	$xt^{-6}xt^3x^{-1}t^3$	$(l=15)$ $(n\geq 13)$

Lemma 3.1. *If $n\leq 100$ and the group $G_n(w)$ is irreducible and is associated with $\mathrm{w}(x,t)$ where $\mathrm{w}(x,t)=(Uj)$ for any j satisfying $1\leq j\leq 21$ then $G_n(w)$ is non-trivial except possibly when $j=12$ and $n=5$ ($w(x,t)=(O15)$ in Appendix B).*

Proof. For (Uj) where $1\leq j\leq 9$ and $16\leq j\leq 21$ we have checked using Magma that $H_n(w)$ maps onto $\mathrm{PSL}(2,q)$ for some $q\geq 5$ whenever n is prime and $5\leq n\leq 97$ (and therefore the corresponding $G_n(w)$ is non-trivial); and for (Uj) where $j\in\{8,9,20\}$, that is, those (Uj) whose associated polynomial is $1-s+s^2-s^3+s^4$, we have also checked that $H_9(w)$ maps onto $\mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,11)$. This is enough to deduce the result for all these values of j. (To see this we use the fact $G_n(w)$ non-trivial implies $G_{kn}(w)$ non-trivial for $k\geqslant 2$. Thus if $G_n(w)$ is non-trivial for n prime and $5\leqslant n\leqslant 100$ it follows that $G_n(w)$ is non-trivial for $5\leqslant n\leqslant 100$ except possibly when n is of the form 2^u3^v. But for $1\leqslant j\leqslant 9$ and $16\leqslant j\leqslant 21$ none of the associated polynomials of the Uj equals 1 so we see from subsection 2.3 that if n is even then $n\equiv\pm 2 \pmod{12}$. This rules out 2^u3^v except for $n\in\{9,27,81\}$ and these values only occur when the associated polynomial is $1-s+s^2-s^3+s^4$. We use a similar argument in the proof of Lemma 11 below but omit the details.)

This leaves $\mathrm{w}(x,t)=(Uj)$ for $10\leq j\leq 15$. Corresponding words w obtained on rewriting these $\mathrm{w}(x,t)$ are the following:

$(\hat{U}10)$	$x_1x_0x_2x_0^{-1}x_1^{-1}x_2^{-2}$;
$(\hat{U}11)$	$x_1x_0^{-1}x_2x_0x_1^{-1}x_2^{-2}$;
$(\hat{U}12)$	$x_0x_2x_1x_2^{-1}x_0^{-1}x_1^{-2}$;
$(\hat{U}13)$	$x_0x_2^{-1}x_1x_2x_0^{-1}x_1^{-2}$;
$(\hat{U}14)$	$x_0x_1x_2x_1^{-1}x_0^{-1}x_2^{-2}$;
$(\hat{U}15)$	$x_0x_1^{-1}x_2x_1x_0^{-1}x_2^{-2}$.

For these examples we now appeal to recent developments in the theory of intersection of Magnus subgroups in one-relator groups [3,14] and an application of this theory to cyclically presented groups [9]. In fact using [3] and [14] it can be shown that for $j \in \{10, 11, 14, 15\}$ the so-called Magnus subgroups $\langle x_0, x_1\rangle$ and $\langle x_1, x_2\rangle$ in the one-relator group $\langle x_0, x_1, x_2 \mid (\hat{U}j)\rangle$ have so-called non-exceptional intersection, that is, $\langle x_0, x_1\rangle \cap \langle x_1, x_2\rangle = \langle x_1\rangle$; that for $j = 12$ we have $\langle x_0, x_1\rangle \cap \langle x_1, x_2\rangle = \langle x_1, x_0^{-1}x_1^2x_0 = x_2x_1x_2^{-1}\rangle$; and that for $j = 13$ we have $\langle x_0, x_1\rangle \cap \langle x_1, x_2\rangle = \langle x_1, x_0^{-1}x_1^2x_0 = x_2^{-1}x_1x_2\rangle$. It then follows immediately from [9, Corollary 1.4] that $G_n\big((\hat{U}j)\big)$ is infinite for $n \geq 6$ and $j \in \{10, 11, 14, 15\}$. Furthermore putting $y_0 = x_2^{-1}, y_1 = x_0^{-1}, y_2 = x_1$ in $(\hat{U}12)$ and $z_0 = x_2, z_1 = x_0^{-1}, z_2 = x_1$ in $(\hat{U}13)$ in both cases yields the group in Section 4.1 of [4] and so $G_n((\hat{U}j))$ is infinite for $n \geqslant 6$ and $j \in \{12, 13\}$. This leaves some cases n where $4 \leq n \leq 5$ and for each of these we have confirmed using **quotpic** that the corresponding group $G_n(w)$ is non-trivial except possibly for the one open case in the statement of the lemma. □

We turn now to what remains, namely the 216 words of A-type. Of these 128 have associated polynomial equal to 1; 69 equal to $1-s+s^2$; 10 equal to $1-s^2+s^4$; and 9 equal to $1-s+s^2-s^3+s^4$.

Lemma 3.2. *If* $\mathrm{w}(x, t)$ *is a word of A–type then, apart possibly from the* 12 *open cases (O16)–(O27) in Appendix B, the associated group* $G_n(w)$ *is either one of (T32)–(T47) in Appendix A or is non-trivial for each value of* $n \leqslant 100$ *for which* $\mathrm{w}(x, t)$ *is valid, that is, satisfies (R1)–(R8) for that value.*

Proof. Of the 128 words $\mathrm{w}(x, t)$ whose associated polynomial equals 1 there are 24 valid for $n \geq 2$, 84 for $n \geq 4$; 16 for $n \geq 6$ and 4 for $n \geq 8$. We remark however that of the 24 valid for $n \geq 2$, although inequivalent for $n > 2$, the number reduces to 16 at $n = 2$; and of the 84 valid for $n \geq 4$, although inequivalent for $n > 4$, the number reduces to 81 at $n = 4$.

For the 108 words valid for either $n \geq 2$ or $n \geq 4$ we have confirmed for 107 words $\mathrm{w}(x, t)$ the associated group $G_n(w)$ to be non-trivial for $n \in \{5, 6, 7, 8, 9\}$; for the one exceptional case $G_n(w)$ is confirmed non-trivial for $n \in \{6, 7, 8, 9, 10, 15, 25\}$ and the case $n = 5$ remains open ((O16) in Appendix B). Therefore for these groups only $n = \{3, 4\}$ and, where appropriate, $n = 2$ remains to be done for $n \leq 100$. In all these remaining cases we confirmed $G_n(w)$ to be non-trivial except for 16 examples of the trivial group ((T32)–(T47) in Appendix A) and 11 more that remain open ((O17)–(O27) in Appendix B). For the 16 words valid for $n \geq 6$ we have confirmed $G_n(w)$ to be non-trivial for $n \in \{6, 7, 8, 9, 10, 15, 25\}$; and for the 4 words valid for $n \geq 8$ we have confirmed $G_n(w)$ to be non-trivial for $n \in \{8, 9, 10, 12, 14, 15, 21, 25, 35, 49\}$.

Of the 69 words $\mathrm{w}(x, t)$ whose associated polynomial equals $1 - s + s^2$ there are 57 valid for $n \geq 5$ and 12 valid for $n \geq 7$. For the first 57 we have confirmed $G_n(w)$ to be non-trivial for $n \in \{5, 7\}$; and for the latter 12 we have confirmed $G_n(w)$ non-trivial for $n \in \{7, 25\}$.

Of the 10 words $\mathrm{w}(x, t)$ whose associated polynomial equals $1 - s^2 + s^4$ there are 3 valid for $n \geq 5$, 5 for $n \geq 7$ and 2 for $n \geq 10$. For the first three we have confirmed $G_n(w)$ non-trivial for $n \in \{5, 7\}$; for the next five $G_n(w)$ is non-trivial for $n \in \{10, 25\}$; and for the last two $G_n(w)$ is non-trivial for $n \in \{10, 14, 25, 35, 49\}$.

Finally, of the 9 words $\mathrm{w}(x, t)$ whose associated polynomial equals $1 - s + s^2 - s^3 + s^4$ there are 8 valid for $n \geq 7$ and 1 valid for $n \geq 9$. For the first 8 we have confirmed $G_n(w)$ non-trivial for $n \in \{7, 9\}$; and for the last word we have confirmed $G_n(w)$ non-trivial for $n \in \{9, 21, 49\}$. □

Putting these results together with Theorem 2.5 we can now state the following theorem.

Theorem 3.3. *Let $G_n(w)$ be an irreducible cyclically presented group whose corresponding word $\mathrm{w}(x,t)$ has length at most 15 and such that $\mathrm{w}(x,t)$ is not equivalent to the word (Oj) for $1 \le j \le 27$. If $n \le 100$ then $G_n(w)$ is trivial if and only if $\mathrm{w}(x,t)$ is equivalent to one of (Ti) where $1 \le i \le 47$.*

Observe that $n \le 5$ for each word (Oj) ($1 \le j \le 27$) and (Ti) ($1 \le i \le 47$), and so we have established Theorem 1.1.

Appendix A: w for which $G_n(w)$ is trivial

(T1)	$x_0^{-1}x_1^{-1}x_2x_0x_1$	$(n=3)$	$(l=10)$
(T2)	$x_0^{-1}x_2x_3^{-1}x_0x_4$	$(n=5)$	$(l=12)$
(T3)	$x_0^{-1}x_1^{-1}x_2^{-1}x_0x_1x_0x_2$	$(n=3)$	$(l=14)$
(T4)	$x_0^{-1}x_1^{-1}x_0x_2^{-1}x_0x_2x_1$	$(n=3)$	$(l=14)$
(T5)	$x_0^{-1}x_1x_2^{-1}x_0x_1^{-1}x_2x_1$	$(n=3)$	$(l=14)$
(T6)	$x_0^{-1}x_1^{-1}x_2x_0x_2^{-1}x_0x_1$	$(n=3)$	$(l=14)$
(T7)	$x_0^{-1}x_2^{-1}x_0x_3x_1$	$(n=5)$	$(l=14)$
(T8)	$x_0^2x_1x_0^{-1}x_1^{-1}$	$(n=2)$	$(l=9)$
(T9)	$x_0^2x_1x_0^{-1}x_1^{-1}$	$(n=3)$	$(l=9)$
(T10)	$x_0^3x_1x_0^{-2}x_1^{-1}$	$(n=2)$	$(l=11)$
(T11)	$x_0^2x_1^2x_0^{-1}x_1^{-2}$	$(n=2)$	$(l=11)$
(T12)	$x_0^2x_1^2x_0^{-1}x_1^{-2}$	$(n=3)$	$(l=11)$
(T13)	$x_0^{-1}x_1^{-1}x_3x_2x_1$	$(n=5)$	$(l=11)$
(T14)	$x_0^4x_1x_0^{-3}x_1^{-1}$	$(n=2)$	$(l=13)$
(T15)	$x_0^3x_1^2x_0^{-2}x_1^{-2}$	$(n=2)$	$(l=13)$
(T16)	$x_0^2x_1^3x_0^{-1}x_1^{-3}$	$(n=2)$	$(l=13)$
(T17)	$x_0^2x_1^3x_0^{-1}x_1^{-3}$	$(n=3)$	$(l=13)$
(T18)	$x_0^{-1}x_1^{-1}x_2^{-1}x_0x_2x_1^2$	$(n=3)$	$(l=13)$
(T19)	$x_0^{-1}x_1^{-1}x_2^{-1}x_0x_2^2x_1$	$(n=3)$	$(l=13)$
(T20)	$x_0^{-1}x_1^{-1}x_2^{-1}x_0^2x_2x_1$	$(n=3)$	$(l=13)$
(T21)	$x_0^{-1}x_2x_1^{-1}x_3x_1$	$(n=5)$	$(l=13)$
(T22)	$x_0^5x_1x_0^{-4}x_1^{-1}$	$(n=2)$	$(l=15)$
(T23)	$x_0^4x_1^2x_0^{-3}x_1^{-2}$	$(n=2)$	$(l=15)$
(T24)	$x_0^2x_1^4x_0^{-1}x_1^{-4}$	$(n=2)$	$(l=15)$
(T25)	$x_0^3x_1^3x_0^{-2}x_1^{-3}$	$(n=2)$	$(l=15)$
(T26)	$x_0^3x_1^2x_0^{-1}x_1^{-1}x_0^{-1}x_1^{-1}$	$(n=2)$	$(l=15)$
(T27)	$x_0^3x_1x_0^{-1}x_1x_0^{-1}x_1^{-2}$	$(n=2)$	$(l=15)$
(T28)	$x_0^4x_1^2x_0^{-4}x_1^{-1}$	$(n=3)$	$(l=15)$
(T29)	$x_0x_2^{-1}x_3^{-1}x_0^{-1}x_3x_2x_1$	$(n=4)$	$(l=15)$
(T30)	$x_0x_1^{-1}x_2x_0^{-1}x_3x_2^{-1}x_1$	$(n=4)$	$(l=15)$
(T31)	$x_0^{-2}x_1^{-1}x_2x_1^2x_0^2x_4^{-1}$	$(n=5)$	$(l=15)$
(T32)	$x_0x_3x_2x_0^{-1}x_3^{-1}$	$(n=4)$	$(l=11)$
(T33)	$x_0^2x_1x_0^{-1}x_1x_0^{-1}x_1^{-1}$	$(n=2)$	$(l=13)$
(T34)	$x_0x_1x_0x_1x_0^{-1}x_1^{-2}$	$(n=2)$	$(l=13)$
(T35)	$x_0x_3x_2^{-1}x_0^{-1}x_2$	$(n=4)$	$(l=13)$
(T36)	$x_0^3x_1x_0^{-1}x_1^{-1}x_0^{-2}x_1$	$(n=2)$	$(l=15)$
(T37)	$x_0^3x_1x_0^{-2}x_1x_0^{-1}x_1^{-1}$	$(n=2)$	$(l=15)$
(T38)	$x_0^2x_1^2x_0x_1^{-1}x_0^{-2}x_1^{-1}$	$(n=2)$	$(l=15)$
(T39)	$x_0^2x_1^2x_0^{-1}x_1x_0^{-1}x_1^{-2}$	$(n=2)$	$(l=15)$
(T40)	$x_0^2x_1x_0x_1x_0^{-2}x_1^{-2}$	$(n=2)$	$(l=15)$
(T41)	$x_0^2x_1x_0x_1^{-2}x_0^{-2}x_1$	$(n=2)$	$(l=15)$
(T42)	$x_0^2x_1x_0x_1^{-1}x_0^{-3}x_1$	$(n=2)$	$(l=15)$
(T43)	$x_0^2x_1x_0^{-1}x_1^2x_0^{-1}x_1^{-2}$	$(n=2)$	$(l=15)$
(T44)	$x_0^2x_1x_0^{-2}x_1x_0x_1^{-2}$	$(n=2)$	$(l=15)$
(T45)	$x_0^2x_1x_0^{-3}x_1x_0x_1^{-1}$	$(n=2)$	$(l=15)$
(T46)	$x_0x_3x_2x_1^{-1}x_2^{-1}x_1x_3^{-1}$	$(n=4)$	$(l=15)$
(T47)	$x_0x_3x_1^{-1}x_2^{-1}x_1x_2x_3^{-1}$	$(n=4)$	$(l=15)$

Appendix B: w(x,t) for which triviality of $G_n(w)$ is open

(O1)	$x^2txtxtx^{-2}tx^{-1}t^{-1}$	$(n=3)$	$(l=12)$
(O2)	$x^3txtxtx^{-3}t^{-1}x^{-1}t$	$(n=3)$	$(l=14)$
(O3)	$xtxtxtx^{-1}txtx^{-1}t^{-1}x^{-1}t^{-1}$	$(n=3)$	$(l=14)$
(O4)	$x^3tx^2t^{-1}x^{-2}tx^{-2}t^{-1}$	$(n=3)$	$(l=13)$
(O5)	$x^2txtx^{-1}tx^{-1}t^{-1}xt^{-1}x^{-1}t^{-1}$	$(n=3)$	$(l=13)$
(O6)	$x^4tx^2t^{-1}x^{-3}tx^{-2}t^{-1}$	$(n=3)$	$(l=15)$
(O7)	$x^3tx^3t^{-1}x^{-2}tx^{-3}t^{-1}$	$(n=3)$	$(l=15)$
(O8)	$x^3txt^{-1}x^{-1}tx^{-1}t^{-1}x^{-2}txt^{-1}$	$(n=3)$	$(l=15)$
(O9)	$x^2txt^{-1}x^{-3}txt^{-1}xtx^{-1}t^{-1}$	$(n=3)$	$(l=15)$
(O10)	$x^2tx^2txt^{-1}x^{-2}t^{-1}x^{-1}t^{-1}x^{-1}t$	$(n=3)$	$(l=15)$
(O11)	$x^2txtx^2tx^{-1}t^{-1}x^{-2}t^{-1}x^{-1}t^{-1}$	$(n=3)$	$(l=15)$
(O12)	$x^2txtx^2tx^{-2}t^{-1}x^{-1}t^{-1}x^{-1}t^{-1}$	$(n=3)$	$(l=15)$
(O13)	$xt^2x^{-1}tx^{-1}tx^{-1}txtxtxt$	$(n=4)$	$(l=15)$
(O14)	$xtxtxtxtx^{-1}t^{-2}x^{-1}t^{-1}x^{-1}t^{-1}$	$(n=5)$	$(l=15)$
(O15)	$x^2txt^{-2}xtx^{-1}t^{-1}x^{-1}t^2x^{-1}t^{-1}$	$(n=5)$	$(l=15)$
(O16)	$x^2txt^{-1}x^{-1}t^{-1}xtx^{-1}t^{-1}x^{-1}t$	$(n=5)$	$(l=13)$
(O17)	$xtxtx^{-1}t^{-1}x^{-1}txt^{-2}$	$(n=4)$	$(l=11)$
(O18)	$xtxt^{-1}x^{-1}t^2xt^{-1}x^{-1}t^{-1}$	$(n=4)$	$(l=11)$
(O19)	$x^2txt^{-1}x^{-1}tx^{-1}t^{-1}x^{-1}t^{-1}xt$	$(n=3)$	$(l=13)$
(O20)	$xtxtx^{-1}t^{-1}xtxt^{-1}x^{-2}t^{-1}$	$(n=3)$	$(l=13)$
(O21)	$x^2txt^{-1}x^{-1}t^{-1}xtx^{-1}t^{-1}x^{-1}t$	$(n=4)$	$(l=13)$
(O22)	$x^3txt^{-1}x^{-1}tx^{-1}t^{-1}x^{-2}t^{-1}xt$	$(n=3)$	$(l=15)$
(O23)	$x^2txt^{-1}x^{-2}txtxt^{-1}x^{-2}t^{-1}$	$(n=3)$	$(l=15)$
(O24)	$x^2tx^2t^{-1}x^{-2}t^2xt^{-1}x^{-2}t^{-1}$	$(n=4)$	$(l=15)$
(O25)	$x^2txt^{-2}x^2tx^{-2}t^{-1}x^{-2}t$	$(n=4)$	$(l=15)$
(O26)	$x^2t^2x^{-1}t^{-1}xtxt^{-2}x^{-1}tx^{-1}t^{-1}$	$(n=4)$	$(l=15)$
(O27)	$xtxtxt^{-1}xt^{-1}x^{-1}tx^{-1}tx^{-1}t^{-2}$	$(n=4)$	$(l=15)$

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