

THE L-FUNCTIONS AND MODULAR FORMS DATABASE PROJECT

JOHN CREMONA

ABSTRACT. The Langlands Programme, formulated by Robert Langlands in the 1960s and since much developed and refined, is a web of interrelated theory and conjectures concerning many objects in number theory, their interconnections, and connections to other fields. At the heart of the Langlands Programme is the concept of an L-function.

The most famous L-function is the Riemann zeta-function, and as well as being ubiquitous in number theory itself, L-functions have applications in mathematical physics and cryptography. Two of the seven Clay Mathematics Million Dollar Millennium Problems, the Riemann Hypothesis and the Birch and Swinnerton-Dyer Conjecture, deal with their properties. Many different mathematical objects are connected in various ways to L-functions, but the study of those objects is highly specialized, and most mathematicians have only a vague idea of the objects outside their specialty and how everything is related. Helping mathematicians to understand these connections was the motivation for the L-functions and Modular Forms Database (LMFDB) project. Its mission is to chart the landscape of L-functions and modular forms in a systematic, comprehensive and concrete fashion. This involves developing their theory, creating and improving algorithms for computing and classifying them, and hence discovering new properties of these functions, and testing fundamental conjectures.

In the lecture I gave a very brief introduction to L-functions for non-experts, and explained and demonstrated how the large collection of data in the LMFDB is organized and displayed, showing the interrelations between linked objects, through our website www.lmfdb.org. I also showed how this has been created by a world-wide open source collaboration, which we hope may become a model for others.

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1. WHAT IS THE LMFDB?

Since the early days of using computers in number theory, computations and tables have played an important part in experimentation, for the purpose of formulating and proving (or disproving) conjectures.

Until the World Wide Web, such tables were hard to use, let alone to make, as they were only available in printed form, or on microfiche! An example relevant for the LMFDB is the 1976 Antwerp IV tables of elliptic curves, published as part of a conference proceedings in Springer Lecture Notes in Mathematics 476, as a computer printout with manual amendments and diagrams.

However, even since the WWW, tables and databases have been scattered among a variety of personal web pages (including my own [Cremona 2015]). To use them, you had to know who to ask, download data, and deal with a wide variety of formats. A few had more sophisticated interfaces, but there was no consistency.

In some areas of number theory, such as elliptic curves, the situation is now much better and easier: packages such as SAGEMATH [Sage 2015], MAGMA [Bosma et al. 1997] and Pari/gp [Pari 2015] contain elliptic curve databases (sometimes as optional add-ons, as they are large). Also, the internet makes accessing even “printed” tables much easier. But the data are still very scattered and incomplete.

The situation is now very much better: we have the **LMFDB!**

The screenshot shows the LMFDB home page with a green header and navigation bar. The main content area is divided into several sections:

- Introduction and more:** Links to Introduction, Features, Tutorial, Map, Contact, Citation, and Future Plans.
- L-functions:** Degree: 1 2 3, and a link to zeros.
- Modular Forms:** Classical, Hilbert, Maass, GL(2), GL(3), Siegel, and Other.
- Varieties:** Elliptic, Curves, /Q, /NumberFields, Genus 2, /Q.
- Fields:** Number fields, Global, Local.
- A Database:** The LMFDB is an extensive database of mathematical objects arising in Number Theory. Sample lists: L-functions, Elliptic curves, Maass forms, Tables of zeros, Number fields.
- Hall of Fame:** Riemann zeta function, Cubic Field of discriminant -23, Ramanujan Δ function and its L-function, First-Rank 4 Elliptic curve and its L-function.
- Search and Browse:** Search for objects with specific properties, or browse categories. Browse: L-functions, Modular forms, Elliptic curves, Number fields.
- Visualize Data:** Explore individual plots or view distributions of various objects. Examples: GL(4) Level one Maass forms, Isogeny graph of elliptic curve 102.c.
- Code and Open Software:** Download the data, download the code, or see how the data were generated. Includes links to GitHub, SageMath, Pari/GP, Magma, and Python.

The LMFDB home page at www.lmfdb.org, January 2016

2. L-FUNCTIONS AND WHY THEY ARE IMPORTANT

L-functions are at the heart of the LMFDB. What are they? We will give a brief survey, referring to number theory textbooks for details.

The simplest L-function is the **Riemann zeta function** $\zeta(s)$. This

- is a *complex analytic function* (apart from a pole at $s = 1$);
- has a *Dirichlet series* expansion over positive integers (valid when $\Re(s) > 1$):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s};$$

- has an *Euler product* expansion over primes p (when $\Re(s) > 1$):

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1};$$

- satisfies a *functional equation*:

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) = \xi(1 - s);$$

- has links to the *distribution of primes*.

2.1. L-functions: a definition. The definition of an L-function encapsulates these properties: it is a **complex function** with a **Dirichlet series** and an **Euler product** expansion which satisfies a **functional equation**. There are other more technical axioms (by Selberg) which we omit here: refer to the LMFDB’s own knowledge database for details: <http://www.lmfdb.org/knowledge/show/lfunction>.

Some of the defining properties have not in fact been proved for all the types of L-function in the database: this can be very hard! For example, Andrew Wiles proved Fermat’s Last Theorem by proving the modularity of certain elliptic curves over \mathbb{Q} , which amounted to showing that the L-functions associated to elliptic curves really are L-functions in the above sense. This is not yet known in general for elliptic curves defined over other algebraic number fields.

Other expected properties of L-functions are not even known for $\zeta(s)$. For example, the **Riemann hypothesis** concerning the the zeros of $\zeta(s)$ has remained open since it was formulated by Riemann in 1859.

2.2. The Riemann Hypothesis. The Riemann Hypothesis states that all the “non-trivial” zeros of $\zeta(s)$ (excluding those coming trivially from poles of $\Gamma(s)$) are on the “critical line” $\Re(s) = 1/2$.

This was (part of) Hilbert’s 8th problem and is also one of the Clay Mathematics Institute Millennium Prize Problems, so a million dollars awaits the person who proves it. There are similar conjectures about

the location of the zeros of all L-functions, which are collectively known as the **Generalized Riemann Hypothesis** (GRH). These are not only of theoretical (or financial!) interest, but have important applications to the complexity of computing important quantities in number theory. For example, computing the class number of a number field is much faster if one assumes GRH for the number field's own L-function, its **Dedekind ζ -function**.

What can a database say in relation to this problem?

It can give the object its own web page (<http://www.lmfdb.org/L/Riemann/>) which shows basic facts about it, and its graph along the critical line $1/2+it$ to “show” the first few zeroes. This is a pedagogical function of the database.

It can also store all the zeroes which have so far been explicitly computed: there are more than 10^{11} (that is *one hundred billion*) of them at <http://www.lmfdb.org/zeros/zeta/>, all computed to 100-bit precision by David Platt (Bristol), who in 2014 won a prize for his contributions to progress on the Goldbach Conjecture. This resource can then be used to study properties of the zeroes, such as their distribution, and connections to random matrices, showing that the database also serves as a research tool.

2.3. Degrees of L-functions. The **Euler product** for a general L-function has the form

$$L(s) = \prod_p 1/P_p(1/p^s)$$

where each $P_p(t)$ is a polynomial, and the product is over all primes p . These polynomials all have the same degree, called the **degree** of the L-function, except for a finite number indexed by primes dividing an integer called the **conductor** of the L-function, where the degree is smaller. The zeros of these polynomials are also restricted in a way depending on another parameter, the **weight**.

For example, $\zeta(s)$ has $P_p(t) = 1 - t$ for all primes p ; the degree is $d = 1$, and the conductor is $N = 1$.

2.4. L-functions of degree 1. There are other L-functions of degree 1, with larger conductor N , which have been studied since the 19th century: **Dirichlet L-functions**. Their Dirichlet coefficients a_n are given by the values of a **Dirichlet character** $a_n = \chi(n)$, meaning that they are **multiplicative** and **periodic** with period N .

An example with $N = 4$ is

$$L(\chi, s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \dots,$$

with all even coefficients 0 and the odd coefficients alternating ± 1 . Dirichlet used such L-functions to prove his celebrated theorem about primes in arithmetic progressions: for any integers $N \geq 1$ and a , there are infinitely many primes $p \equiv a \pmod{N}$, provided that a and N are coprime. The previous example can be used not only to show that there are infinitely many primes $p \equiv 1 \pmod{4}$ (for which $\chi(p) = +1$) and infinitely many primes $p \equiv 3 \pmod{4}$ (for which $\chi(p) = -1$), but also to show that (in a precise sense) the primes are *equally distributed* between these two classes.

This is a *complete* list of all L-functions of degree 1. For degrees greater than 1, a complete classification has not yet been established, though a wide variety of sources of L-functions is known, and in some cases (such as in degree 2, see below) we conjecture that all L-functions do arise from these known sources.

2.5. Other sources of L-functions. A wide variety of mathematical objects have L-functions: algebraic number fields, algebraic varieties (including curves). There is a general term **motive** for objects which have L-functions.

In many cases, while we know how to define the L-function of a more complicated object, it has not yet proved that it actually satisfies the defining axioms for L-functions. Even for elliptic curves over \mathbb{Q} , this would have been true until the mid-1990s; for elliptic curves over real quadratic fields such as $\mathbb{Q}(\sqrt{2})$ it was true until 2013! Now, these elliptic curves are known to be modular [Siksek et al. 2015].

2.6. L-functions of number fields. An **algebraic number field**, or simply **number field**, is a finite extension of the rational field \mathbb{Q} , such as $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(i)$ or $\mathbb{Q}(e^{2\pi i/m})$. Every number field K has an L-function called its **Dedekind zeta function** $\zeta_K(s)$, defined in a similar way to Riemann's $\zeta(s) = \zeta_{\mathbb{Q}}(s)$, and with similar analytic properties.

Just as the analytic properties of $\zeta(s)$ imply facts about the distribution of primes, from the analytic properties of $\zeta_K(s)$ we can deduce statements about prime factorizations in the field K . For example, taking $K = \mathbb{Q}(e^{2\pi i/m})$ we can prove Dirichlet's Theorem on primes in arithmetic progressions using a combination of algebraic and analytic properties of $\zeta_K(s)$.

Also, just as some properties of $\zeta(s)$ are not yet proved (for example the Riemann Hypothesis), the same is true for $\zeta_K(s)$: the Generalized Riemann Hypothesis or GRH remains unsolved.

2.7. L-functions of curves. Algebraic **curves** defined over algebraic number fields also have L-functions, whose degree depends on both the

degree of the field over which the curve is defined, and the genus of the curve. So an elliptic curve over \mathbb{Q} , which is a curve of genus 1 defined over a field of degree 1, has a degree 2 L-function, elliptic curves over fields of degree d have L-functions of degree $2d$, and so on.

It is widely believed that all degree 2 L-functions arise as follows: they are either products of two degree 1 L-functions, or come from elliptic curves over \mathbb{Q} , or from (a special kind of) **modular form**. The insight of Weil, Taniyama, Shimura and others in the 1960s and 1970s was to realize that the latter two sources actually produce the *same* L-functions! This insight is behind the famous theorem of Wiles *et al.* that “every elliptic curve (over \mathbb{Q}) is modular”, from which Fermat’s Last Theorem was a consequence. But it is still an unsolved problem to show that those degree 2 L-functions which are not products of degree 1 L-functions do all arise from automorphic forms.

2.8. Higher degree L-functions. For degrees 3 and 4, we do not yet even have a conjecture concerning all sources of L-functions, and for those which are known, not all the conjectured connections between them have been proved.

We mentioned above the recent result [Siksek et al. 2015] that elliptic curves defined over real quadratic fields (such as $\mathbb{Q}(\sqrt{5})$) are modular. This means that two sources of L-functions of degree 4: on the one hand, elliptic curves over such a field, and on the other hand **Hilbert modular forms** over the same field, actually produce the *same* L-functions. Such results are extremely deep and require a vast amount of theory to establish, including real, complex and p -adic analysis and algebra, as well as some explicit computations (the ArXiv version of [Siksek et al. 2015] includes a number of MAGMA scripts).

By contrast, over *imaginary* quadratic fields (e.g. $\mathbb{Q}(\sqrt{-1})$) we conjecture, but cannot prove in general, that elliptic curves have L-functions also attached to a different kind of modular form, **Bianchi modular forms**. These can be computed, and work is in progress in entering many examples into the LMFDB, even though they are not all known to “be modular” and hence have genuine L-functions.

Modularity of *individual* elliptic curves over imaginary quadratic fields can be proved using the Serre-Faltings-Livné method (which uses Galois representations rather than analysis) as explained in a 2008 paper [Dieulefait et al. 2010] by Dieulefait, Guerberoff and Pacetti. We are currently using their method to prove modularity of *all* the curves in the database; at the same time we are developing enhancements to the algorithm to make it more efficient. A theoretical proof that all elliptic curves over these fields are modular seems very far off, so even

in the world of L-functions of degree 4 it is still important to carry out experiments and collect data.

2.9. Showing connections through the LMFDB. The LMFDB shows connections between different objects with the same L-function, such as those described above, by linking its databases of (for example) elliptic curves over real quadratic fields, and Hilbert modular forms over the same field. The home page of each elliptic curve includes a link to the associated Hilbert modular form, and to the associated L-function, and (in progress) vice versa.

One difficulty we have encountered in setting up these links on the website, which is perhaps typical in a large project where many different individuals are providing data, is to maintain *consistency of labelling* of objects. Over the field $\mathbb{Q}(\sqrt{5})$, the Hilbert modular forms were computed (in MAGMA) by John Voight (Dartmouth College) and Steve Donnelly (Sydney) [Dembélé and Voight 2013], while the elliptic curves were computed (in SAGEMATH) by Jonathan Bober (Bristol), William Stein (Washington), Alyson Deines (CCR) and others [Bober et al. 2013]. These groups used essentially the same naming convention, but we were careful to check that the labels of matching objects did match exactly, resulting in one set of data (the elliptic curves) requiring relabelling.

3. THE LMFDB DATABASE

The LMFDB consists of both a database, where the data collection itself is organized and stored, together with the website www.lmfdb.org. This provides a sophisticated user interface to the data, has home pages for individual objects in the database, showing links between related objects, and also provides an online repository of knowledge about L-functions and related objects, through its **knowledge database**.

Both database and website are currently hosted on servers at Warwick, funded by EPSRC; until 2013 they were hosted at the University of Washington on NSF-funded servers administered by William Stein. Plans are also underway to have mirror sites in other countries: this is an international project.

The LMFDB is also a group of mathematicians who collaborate to create and develop the database and its website. We will say more about this collaboration in the final section.

3.1. The database and website software. We are using the open-source database software `MongoDB`. This currently holds nearly a terabyte of data and indices. This choice was made because `MongoDB` allows data to be organized in a completely flexible schema, rather than having to specify the schema for each item in advance as with SQL databases. It also has a powerful `Python` interface, `PyMongo`, which suits the project well, since it allows the website code to use other `Python` modules such as `Flask` (a web framework), and to have access to all the power of `SAGEMATH`, another large `Python`-based open source mathematical software project. All of these are open source, which is another essential requirement. (Note, however, that not all of the data in the database have been computed using open source tools.) The website code is a collaborative open source project hosted at `GitHub` (see <https://github.com/LMFDB/lmfdb>).

Basing all the website code on `Python` has many advantages. It is relatively easy to learn to use, which is important since we want the barriers to new people joining our project to be as low as possible. And it is phenomenally powerful, giving access to a vast array of additional modules for interfacing with the database (`PyMongo`), running the web framework (`Flask`), web page templating (`Jinja`), testing and more.

Anyone contributing to the project who wants to do more than just donate data has to learn how to use this software. At project workshops we run tutorial sessions for newcomers, where code is written by beginners under the guidance of more experienced peers. All code is reviewed and tested before being adopted, as well as being subject to some automated testing.

3.2. Database organization. The database as a whole consists of around 35 individual databases containing collections of mathematical objects (including `elliptic_curves`, `hilbert_modular_forms`, and `number_fields`) and other data such as the `knowledge` database, which holds the contents of `knowls` (see below).

The data are indexed in various ways for faster searching, and, of course, backed up regularly. Many parts of the database can also be recreated from plain text data files which are stored in separate `Git` repositories, also hosted on `GitHub`.

Each constituent database contains collections of records, and these records hold the data in a flexible format: additional data fields can be added later.

3.3. Sample database entry. To take just one example, the database `number_fields` contains just one collection `fields`, for which a typical

entry looks as follows (after being converted by PyMongo into a Python dictionary):

```
{u'_id': ObjectId('4cb80fdb5009fb52db0946b6'),
  u'class_group': u'',
  u'class_number': 1,
  u'coeffs': u'1,0,-1,1',
  u'degree': 3,
  u'disc_abs_key': u'00123',
  u'disc_sign': -1,
  u'galois': {u'n': 3, u't': 2},
  u'label': u'3.1.23.1',
  u'ramps': [u'23'],
  u'signature': u'1,1',
  u'unitsGmodule': [[3, 1]]}
```

Here we see the coefficients of the minimal polynomial $x^3 - x^2 + 1$ of a generator of the field stored as `coeffs`, and the label '3.1.23.1' which also uniquely determines the field. Invariants of the field which are easy to compute on the fly, and to which we do not need to provide direct access through database queries, need not be stored, while quantities which might be expensive to compute, or for which we may want to run searches, are stored and indexed.

This is only a simple example. The database entry for an individual elliptic curve over \mathbb{Q} currently contains 33 fields, some very technical. The number of fields grows over time as new data are contributed. For example, in 2015 Jeremy Rouse offered to provide information concerning the 2-adic Galois Representation attached to every elliptic curve over \mathbb{Q} , after developing and implementing an algorithm to determine this jointly with David Zureick-Brown (see [Rouse et al. 2015]). He provided us with a MAGMA script of their implementation, we ran it and uploaded the data, and added a corresponding section on the home page of every curve showing these additional data.

3.4. Software choices, pros and cons. Using off-the-shelf software has plenty of advantages but will never be perfect for a mathematical project.

Most mathematicians, even those with substantial computational experience and expertise, know almost nothing about databases or running websites, and many of the contributors to the LMFDB knew nothing at all about these before they joined the project. Decisions about the specific software used by the project was made by those who did

have such experience, notably William Stein (lead developer of SAGEMATH) and Harald Schilly (another key developer of the SAGEMATH-CLOUD project, <https://cloud.sagemath.com/>).

We have already seen some of the advantages of our choice of database, MongoDB. There are disadvantages too: MongoDB data consists of strings or integers or floating point values, with strings as keys. Values can also be lists of these, but a serious deficiency for number-theoretic data is that the integers cannot be larger than 2^{32} . This means that most data fields which hold integers have to be stored as strings, and this limits functionality, such as searching for the value being in a certain range.

Similarly, rational numbers cannot be stored as such, or even as a pair of integers [*numerator, denominator*] if these could be large, so instead they are stored as strings such as ‘1728’ or ‘-122023936/161051’. Building on these, considerable thought has to be given as to how to store more complicated data, such as an element of a number field. Decisions such as these are made by consensus at LMFDB workshops, since they affect all developers, even though the effects of such decisions are hidden from users of the website.

4. THE LMFDB WEBSITE

The LMFDB website serves several purposes. It provides

- a shop window for the data;
- a way to visualize the data, and the connections between different, linked mathematical objects;
- a way to browse types of object;
- a way to search for objects with specified properties;
- a repository of knowledge through its “knowledge database”;
- a source of data for downloading for further work.

Catering for several different audiences at once is hard to get right!

4.1. Technical support (or lack of). The project would benefit greatly from having technical support staff. Our current grant from EPSRC does not provide this—it does support six postdoctoral researchers, who all have a certain amount of experience writing mathematical software, but not any dedicated software engineers. For this, we are currently relying on charitable contributions of time. We would not be where we are now, and indeed the website would never have been launched, without the enormous contributions of one person in particular: Harald Schilly, a doctoral student in Vienna and software

consultant, who knows more than the rest put together about Python, MongoDB, Flask, and the rest.

From September 2015, through the Horizon 2020 European Research Infrastructure project OpenDreamKit (<http://opendreamkit.org/>), which provides substantial funding for the development of open source computational mathematics, we are currently seeking to employ a software engineer to provide support to the project.

4.2. Homepages. A key organizing principle of the LMFDB is that every object has its own homepage. These have mathematically meaningful, permanent URLs which follow a carefully thought out schema. The home pages themselves are created on demand from templates, filled in with data partly retrieved directly from the database and partly computed on the fly. For example, the elliptic curve with label 5077a1 has URL <http://www.lmfdb.org/EllipticCurve/Q/5077/a/1>.

Each homepage gives a view of the object (depending on its nature), highlighting its most important properties, with *breadcrumbs* to show its position in the whole. In some cases, where the object has some interesting additional historical or mathematical significance, this can also be shown on its home page. For example, the elliptic curve 5077a1 was used by Dorian Goldfeld in 1985 to solve Gauss's class number problem effectively, by making use of a new connection between the problem and L-functions of elliptic curves, and this piece of historical information is shown on the curve's home page.

A **related objects** box on each homepage provides links between related objects. For example, from the pages of an elliptic curve, or a number field, or a modular form there are links to the associated L-function.

Where possible, on the home page of an object, we make it possible to see and download code which will re-create the object in one of the standard number-theoretical packages (SAGEMATH, Pari/gp or MAGMA) and work with it there. In this way, the LMFDB can be used by students learning a subject who wish to work out their own examples, as well as researchers wishing to carry out larger-scale investigation starting from the LMFDB data. A more sophisticated programming interface through SAGEMATH is also planned.

4.3. Searching and browsing. Each class of objects in the LMFDB has its own **Browse and Search** page.

The **Browse** section is intended to be usable by people who know nothing of the underlying theory but want to browse through examples without having to type anything or have technical knowledge.

The **Search** section is more for experts looking for a specific object (possibly by its label), or for an object with certain properties: “a number field with Galois group C_5 ramified only at $p = 5$ ”, or “an elliptic curve with rank 2 and non-trivial Tate-Shafarevich group”, or “a classical modular form of weight 12 and level 12”. This leads to a **Search Results** page listing all database entries which match (if any), with links to the home pages of each individual matching object.

4.4. Knowledge and knowls. The knowledge aspect of the LMFDB exists in the first place as a glossary of technical terms used on the web pages, so the pages themselves do not get cluttered up, and there is consistency between pages on basic definitions.

The mechanism which serves these is the **knowl**, created by Harald Schilly and first demonstrated at an LMFDB workshop. The text expands within the page and can be dismissed after reading, without any need for “pop-ups” or new pages.

Knowls can be used anywhere on the web—for example, I use them on my own web page of preprints and publications to display abstracts of papers. Another good example of their use is in the online undergraduate textbook on Linear Algebra by Robert Beezer [Beezer 2014]. For more about knowls and how to use them, see the knowl on the LMFDB itself, <http://www.lmfdb.org/knowledge/show/doc.knowl>, or the page <http://aimath.org/knowledge/>.

The content of knowls can be edited by any project member (someone who has a login account), and is itself stored in the database.

5. THE LMFDB PROJECT

5.1. The LMFDB as a collaborative project. The LMFDB was first conceived at an AIM workshop in 2007. It holds regular workshops, which are run along the lines of AIM workshops: few talks and a lot of hard work. As well as individual workshops of around 30 people, there are smaller groups who meet to work together on specific projects, and there have also been longer periods of activity hosted at MSRI, during the semester programme “Arithmetic Statistics” in 2011, and ICERM, during the semester programme “Computational Aspects of the Langlands Program” in late 2015 (see <http://icerm.brown.edu/sp-f15/>). All members of the organizing committee for the latter are LMFDB contributors, and we expect that the LMFDB will make a substantial leap forward during the semester.

The AIM connection remains strong: both Brian Conrey (Director of AIM) and David Farmer (Director of Programs at AIM) are number

theorists who have been intimately connected with the project from the start.

We have Editorial and Management Boards, but essentially all decisions are made by consensus at workshops.

5.2. Funding. During 2008–2012, the LMFDB was funded by NSF FRG grant DMS:0757627; currently (2013–2019) it is supported by Programme Grant EP/K034383/1 from the UK research council EPSRC. The investigators on this are the author and Samir Siksek (Warwick) and Brian Conrey (AIM and Bristol), and Andy Booker and Jon Keating (Bristol). David Farmer (AIM) is a project partner, as are Fernando Rodriguez-Villegas (ICTP), William Stein (Washington) and Mike Rubinstein (Waterloo).

These research grants provide funding both for LMFDB workshops and for servers hosting the database and website; the NSF FRG grant also paid for some technical software support.

5.3. Collaboration. The LMFDB encompasses such a wide range of mathematics, and it is essential to have an equally wide range of mathematical expertise contributing to the project. Many of the collaborators on the LMFDB project, who are all listed at <http://www.lmfdb.org/acknowledgment>, have contributed not by coding for the website but by providing the data (without which the project would be nothing!). More contributors are always welcome.

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WARWICK MATHEMATICS INSTITUTE, ZEEMAN BUILDING, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UK