MODULAR SYMBOLS

by

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ABSTRACT

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In this thesis, an algorithm is given for computing certain spaces of automorphic forms defined over Euclidean Complex Quadratic Fields, by means of calculating explicitly the action of the Hecke algebra on the first rational homology group of hyperbolic upper half-space modulo a discrete subgroup, which is a congruence subgroup of $SL_2(O_K)$ where K is the field in question. The motivation for this is to provide evidence for a precise conjecture, similar to Weil's conjecture over the rationals, relating certain of these forms with elliptic curves defined over K.

Extensive tables are given of the results of implementing the algorithms on a computer, giving the dimension of the space of cusp forms of weight two for $\Gamma_0(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of O_K with norm less than a certain bound, together with the splitting of the space into eigenspaces for the Hecke algebra, and the first few Hecke eigenvalues for the newforms at each level. We also give tables of elliptic curves defined over K, with small conductor, with the Trace of Frobenius at the first few primes: here the conductor was found by implementing Tate's algorithm on the computer. The curves were found by a systematic search procedure.

Lastly, we explain certain connections within the tables by proving some results by means of an extension of the theory of Atkin-Lehner to the present situation. In particular, we show that newforms always occur in pairs, with opposite eigenvalues for a certain involution, not necessarily at the same level.

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<u>CONTENTS</u>

INTRODUCTION

The aim of this thesis is to provide evidence for an analogue of Weil's conjecture, that every elliptic curve defined over $\mathbb{Q}$ is parametrized by modular functions, for complex quadratic fields. A large amount of evidence exists in support of the conjecture in the rational case: see, for example, the tables in Volume IV of the Antwerp Proceedings [2]; also, the conjecture has been proved in certain special cases: for example, in the case of elliptic curves over $\mathbb{Q}$ with complex multiplication. Before one can even state a conjecture for complex quadratic fields, one must define what is meant by a 'modular function' over such a field, and in order to collect evidence for the conjecture, one has to have a method of calculating these objects in particular cases. Indeed, it was only after a substantial amount of data had been collected that the precise nature of the relation between automorphic forms over the fields considered and elliptic curves over the same fields began to emerge, although such a connection is predicted as part of the general 'Langlands Philosophy'.

Most of the work behind this thesis was concerned with developing a method of calculating automorphic forms over a complex quadratic field, and with carrying out the computations for the Euclidean fields.

To set the scene, and for later reference, we start by presenting a brief summary of the theory of modular forms and elliptic curves over the rational field $\mathbb{Q}$. This is the content of Chapter 1. No attempt has been made to be complete in this survey: only those aspects of the rational theory which will be referred to later have been included. For a more comprehensive survey, see the article by Birch and Swinnerton Dyer in Antwerp IV (op.cit.).

Chapter 2 is spent in developing the geometry of hyperbolic upper half-space, and in particular the action of certain discrete groups thereon. The main references here are Beardon [4] and Swan [20].

Then, in Chapter 3, certain functions on this space are introduced, which will play the rôle of the 'cusp forms of weight 2' of the rational theory. Our approach here largely follows that of Weil in [24], but avoids the use of adèles and is more elementary in nature. As in the rational case, one calculates the cusp forms of weight 2 by means of the action of the Hecke algebra on homology: the heart of the thesis lies in Chapter 4, where it is shown how to generalize modular symbols in order to provide an algorithm for computing the homology of upper half-space modulo a congruence subgroup, and the Hecke action on homology. The algorithm is given explicitly for each of the five Euclidean fields, with some remarks on how one would proceed with the others. It is a generalization of an algorithm given in Manin [10] for the rational case. While the derivation of the algorithm is highly geometric, the end result is purely a question of algebraic manipulation of arithmetic symbols, and is suitable for implementation on a computer.

Chapter 5 consists largely of tables of the results of the computations: for each of the Euclidean fields, and for each ideal $\mathfrak{a}$ in such a field with $N\mathfrak{a}$ less than some bound, we give the dimension of the space of cusp forms of weight 2 for $\Gamma_0(\mathfrak{a})$, with a list of the first few Hecke eigenvalues of the newforms at each level. Also included are tables of elliptic curves for each field with small conductor, found by a search procedure. Some remarks are made on the computations themselves, as well as a discussion of the results in the tables. Certain of these results were already available in work of Mennicke and Grunewald [11], [12], who use a slightly different approach.

In Chapter 6 some of the patterns and regularities in the tables of the previous Chapter are explained by means of certain 'twisting operators'. These are straightforward generalizations of the R_q operators of Atkin-Lehner in [3]: for example, one finds that newforms always occur in pairs, not usually at the same level, with opposite eigenvalues for a certain involution, and that only one of each pair seems to correspond to an elliptic curve. Also in this Chapter is a development of the theory of Atkin-Lehner, generalized to a complex quadratic field; some of this has already been provided by Miyake [13] in the more general context of automorphic forms over a general global field.

CHAPTER 1

Modular Forms and Elliptic Curves over Q

This Chapter is a survey of the theory of Modular Forms and Elliptic Curves over the Rationals. We have restricted our attention to those aspects of the theory which will be referred to later. For the theory of Modular Forms, the main references are the books by Lang [9] and Shimura [17]; for Elliptic Curves, see Tate's article [21]; for the connections between them, see Antwerp IV [2], and especially the article there by Birch and Swinnerton Dyer.

In the last section we describe the contents of the remaining chapters of the thesis.

§1.1 Geometry of the upper half-plane

The group $GL^+(2,R)$ of real 2×2 matrices with positive determinant acts on the upper half-plane

$$H \;:=\; \{z \in C : \mathrm{Im}(z) > 0\}$$

according to the familiar rule

$$(1.1.1) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) \;=\; \frac{az+b}{cz+d}.$$

Extend H by adding the point $i\infty$ at infinity, and the real line, to obtain $H^* := H \cup R \cup \{i\infty\}$; then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends $\frac{-d}{c}$ to $i\infty$ and $i\infty$ to $\frac{a}{c}$ (if $c = 0$ then both of these quotients are to be interpreted as $i\infty$). The extended upper half-plane H^* is given the standard topology (c.f. Shimura [17] §1.5), and a metric $ds^2 = (dx^2 + dy^2)/y^2$, where $z = x + iy$, with respect to which its geometry is hyperbolic, and $GL^+(2,R)$ is its group of isometries.

Next consider the action of a discrete subgroup of $GL^+(2,R)$, for instance $\Gamma = SL(2,Z)$; here 'discrete' means discrete in the matrix topology, where $GL^+(2,R)$ is identified with a subset of R^4 in the obvious way. It is a fact (c.f. Beardon [4] Theorem 4.2) that a subgroup of $GL^+(2,R)$ is discrete if and only if its action on H^* is discontinuous, in the sense that every compact subset of H^* meets only a finite number of its images under the action. A fundamental region for the action of Γ is given by the triangle F with vertices at $i\infty$, ρ and ω (where $\rho = \exp(\pi i/3)$ and $\omega = \rho^2$); the two vertical sides are identified by the action of $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which takes z to z+1; the bottom, a circular arc from ρ through i to ω, is self-identified by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which sends z to $-1/z$ and thus fixes i while interchanging ρ and ω. The orbit of $i\infty$ under Γ consists of $\{i\infty\} \cup \mathbf{Q}$; these points are called <u>cusps</u>. The point at infinity is called $i\infty$ in this context to emphasize that as $z = x + iy$ approaches the cusp in F, the real part x is bounded, while the imaginary part y tends to infinity.

If G is a subgroup of finite index k in Γ, and $\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$ is a set of right coset representatives for G in Γ, then $\bigcup_{j=1}^{k} \alpha_j F$ is a fundamental region for G in H^*: it is a (hyperbolic) polygon with a certain number of edges identified in pairs by elements of G. The quotient space $X_G(\mathbf{C}) := G\backslash H^*$ can be given a complex structure (c.f. Shimura [17] §1.5), with respect to which it is a compact Riemann surface. In particular the local variable at $i\infty$ is $e^{2\pi iz/h}$, where h is the smallest positive integer such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ belongs to G. There is a natural projection $\varphi: H^* \to X_G(\mathbf{C})$. The only points of H which are fixed by nontrivial elements of Γ are those in the orbits of i and ρ (which are fixed by S and TS, of orders 2 and 3 respectively); the images of these orbits under φ are called <u>elliptic points</u> of $X_G(\mathbf{C})$; the images of $\mathbf{Q} \cup \{i\infty\}$ are called <u>parabolic points</u>, or <u>cusps</u>; both sets are finite, and φ is unramified outside them.

§1.2 Congruence subgroups

For a given positive integer N we define the following subgroups of Γ:

$$\Gamma(N) \;:=\; \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1,\ b \equiv c \equiv 0 \pmod{N}\right\};$$

$$\Gamma_0(N) \;:=\; \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N}\right\}.$$

Note that $\Gamma_0(N) \supset \Gamma(N)$; and a group G with $\Gamma(N) \subset G \subset \Gamma$ is called a congruence subgroup of level N; in general, a congruence subgroup is one which contains $\Gamma(N)$ for some N. A short computation shows (c.f. Shimura pp. 20-24) that

$$[\,\Gamma : \Gamma(N)\,] \;=\; N^3 \prod (1-p^{-2})$$

and

$$[\,\Gamma : \Gamma_0(N)\,] \;=\; N \prod (1+p^{-1})$$

where in each case the product is taken over the distinct primes dividing N.

Write $X_N(\mathbb{C})$ for $X_{\Gamma_0(N)}(\mathbb{C})$. There is a smooth projective curve, denoted $X_0(N)$, such that $X_N(\mathbb{C})$ is precisely the set of complex points on $X_0(N)$.

§1.3 Modular Forms

Consider functions $f\colon \mathbb{H} \to \mathbb{C}$. If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $GL^+(2,\mathbb{R})$, and k is a non-negative even integer, we define a new function $(f|\alpha)_k$ by

(1.3.1) $$(f|\alpha)_k(z) \;:=\; f(\alpha z)(ad-bc)^{\frac{1}{2}k}(cz+d)^{-k}.$$

Let G be a subgroup of Γ of finite index.

Definition 1.3.2 *A modular form of weight k for G is a function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying*

(i) f is meromorphic on $\mathbb{H}$;

(ii) $(f|\alpha)_k = f$ for all $\alpha \in G$;

(iii) f is meromorphic at every cusp of $G\backslash\mathbb{H}^*$.

For the cusp $i\infty$, condition (iii) means the following. Because of condition (ii), f is invariant under $z \to z+h$ where h is the width of the cusp at $i\infty$ as in §1.1, and we can thus write $f(z) = F(e^{2\pi i z/h})$ for some function $F(q)$, meromorphic in the domain $0 < |q| < r$ for some $r>0$. Condition (iii) means that F is meromorphic at $q = 0$. The condition at other cusps can be reduced to this case by using a suitable element of Γ to map the cusp to $i\infty$. For more detail, see

Shimura (op. cit. §2.1).

A modular form of weight 0 for G is a modular <u>function</u> for G, and can be written as $\varphi\circ g$ for some function g on $X_G(\mathbf{C})$, where φ is the projection of §1.1.

A modular form of weight 2 for G satisfies

(1.3.3) $$f(\alpha z)d(\alpha z) = f(z)dz$$

for all $\alpha \in G$, by (1.3.1) and (1.3.2)(ii); so f(z)dz is an invariant differential and can be written as $\varphi\circ\,\omega$ for some differential ω on $X_G(\mathbf{C})$. We will mainly be concerned with forms of weight 2, and will frequently omit the subscript 2 and write $f|\alpha$ for $(f|\alpha)_2$.

A modular form f(z) is called a <u>cusp form</u> if the associated function F(q) vanishes at q = 0. Denote the space of cusp forms of weight k for G by $S_k(G)$. In particular for $G = \Gamma_0(N)$ we may take h = 1 since $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \in \Gamma_0(N)$ for every N. So a cusp form of weight 2 for $\Gamma_0(N)$ is a holomorphic function $f\colon \mathbf{H} \to \mathbf{C}$ such that

(i) $f(\alpha z)d(\alpha z) = f(z)dz$ for all $\alpha \in \Gamma_0(N)$;

(ii) f(z) has a Fourier expansion $\Sigma_{n=1}^{\infty} a_n q^n$, where $q = e^{2\pi iz}$, and $a_n \in \mathbf{C}$.

This sets up a one-one correspondence between cusp forms of weight 2 for G and regular differentials on $X_G(\mathbf{C})$.

Using the algebraic and complex structures on $X_G(\mathbf{C})$ one can compute a formula for the dimension of $S_k(G)$ for various subgroups G of Γ: see Shimura §2.6.

The definition in (1.3.1) of the action of $GL^+(2,\mathbf{R})$ on forms of weight k can be extended by linearity to the real group ring of $GL^+(2,\mathbf{R})$; explicitly, if $r_j \in \mathbf{R}$ and $\alpha_j \in GL^+(2,\mathbf{R})$, then

(1.3.4) $$(f|\,\Sigma\, r_j\alpha_j)_k \;:=\; \Sigma\, r_j(f|\alpha_j)_k.$$

We can now define some particular elements of this group ring: for any positive integer m,

(1.3.5) $$T_m \;:=\; \sum_{\substack{d|m\\ d>0}}\;\sum_{b \bmod d} \begin{pmatrix} n/d & b\\ 0 & d\end{pmatrix}.$$

The matrices appearing in the above sum form a set of right coset representatives for Γ in the set of all integer matrices of determinant m. In particular, when $m=p$, a prime, we have

$$(1.3.6) \qquad T_p = \sum_{a=0}^{p-1} \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$(f|T_p)_k(z) = p^{-\frac{1}{2}k} \Sigma\, f\left(\frac{z+a}{p}\right) + p^{\frac{1}{2}k} f(pz).$$

Substituting in the Fourier expansion $f(z) = \Sigma a_n q^n$ (where $q = e^{2\pi i z}$) we find that

$$(1.3.7) \qquad (f|T_p)_k(z) = p^{\frac{1}{2}k} \sum_{n=1}^{\infty} (a_{np} + p.a_{n/p}).q^n$$

where we interpret $a_{n/p}$ as 0 when p does not divide n.

The Hecke algebra is the algebra generated by all the T_m; it preserves $S_k(\Gamma)$, and has the following properties.

(i) $T_m T_n = T_{mn}$ if m and n are relatively prime;

(ii) $T_{p^r} T_p = T_{p^{r+1}} + p.\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}.T_{p^{r-1}}$ if p is prime;

(iii) The algebra generated by the T_m is also generated by the T_p for p prime, and is commutative;

(iv) There is a formal 'Euler product' identity

$$\sum_{n=1}^{\infty} T_n.n^{-s} = \prod_{p \text{ prime}} (I - T_p.p^{-s} + \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}.p^{1-2s})^{-1};$$

(v) If $f(z) = \Sigma a_n q^n$ is an eigenform for all the Hecke operators, say $T_n f = \lambda_n f$ for $n > 1$, then $a_1 \neq 0$, and if f is normalized so that $a_1 = 1$ then $a_n = \lambda_n$ for all n;

(vi) There exists a scalar product (the 'Petersson inner product') on $S_k(\Gamma)$ with respect to which the Hecke operators are Hermitian, and hence there is a basis of $S_k(\Gamma)$ consisting of forms which are eigenforms for all the Hecke operators.

The Hecke action on $S_k(G)$ for proper subgroups G of Γ is more complicated to describe. In the case of $G = \Gamma_0(N)$, write S_N for $S_k(G)$; then for $(m,N) = 1$, the operator T_m takes S_N to itself, and for the smaller set $\{T_m : (m,N) = 1\}$ properties (i) - (vi) still hold with obvious modifications.

Another important operator is induced by $W = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ which on H

sends $z \to -1/Nz$; this matrix normalizes $\Gamma_0(N)$ and so induces a transformation of $X_N(\mathbf{C})$ which is an involution since W^2, which is $\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$, acts trivially; it commutes with all the Hecke operators.

Secondly, complex conjugation is also an involution of $X_N(\mathbf{C})$, corresponding to the transformation $z \to -\bar{z}$ (reflection in the imaginary axis) on H^*, since the conjugate of $e^{2\pi iz}$ is $e^{2\pi i(-\bar{z})}$. Conjugation commutes with W and all the Hecke operators, so that there is a basis for S_N consisting of eigenforms for all the T_p (for $p \nmid N$), for W, and for conjugation.

If $f(z) \in S_M$ for some M dividing N, then the function $z \to f(kz)$ belongs to S_N for any number k dividing the quotient $\frac{N}{M}$. Such forms are called <u>oldforms</u>; an eigenform which is not an oldform is called a <u>newform</u>; it will be orthogonal to all the oldforms with respect to the Petersson inner product.

§1.4 Elliptic Curves: their Zeta Functions and Conductors

Let E be an elliptic curve (an irreducible non-singular algebraic curve of genus 1, with a distinguished point) defined over $\mathbf{Q}$, with equation

(1.4.1) $$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

(where the point at infinity is the distinguished point). Assume that the coefficients a_i are (rational) integers, and that the discriminant Δ is as small as possible under this condition. (The discriminant is given by a polynomial in the a_i, and its non-vanishing is a necessary and sufficient condition for E to be non-singular; for a formula, see Tate's article in [2]). For a prime p, we can reduce the coefficients a_i modulo p to obtain a curve $\tilde{E}_p$ over $\mathbf{GF}(p)$. If p does not divide Δ then $\tilde{E}_p$ is also an elliptic curve, and we may define its L-series as

$$L(\tilde{E}_p, u) := ((1 - \alpha_1 u)(1 - \alpha_2 u))^{-1}$$

where α_1 and α_2 are the characteristic roots of the Frobenius map $(x,y) \to (x^p, y^p)$, which is an endomorphism of $\tilde{E}_p$, and satisfy

(i) $\alpha_1\alpha_2 = p$; $|\alpha_1| = |\alpha_2| = p^{\frac{1}{2}}$;

(ii) $\alpha_1 + \alpha_2 = 1 + p - M$ (the 'trace of Frobenius');

here M is the number of points on $\tilde{E}_p$ (this is one more than the number of solutions of (1.4.1) modulo p, because of the point at infinity).

On the other hand, if p does divide Δ, then $\tilde{E}_p$ has a singular point P, defined over GF(p). There are three possibilities for P:

(i) P is a double point with each of the tangent directions defined over GF(p): then define $L(\tilde{E}_p,u) := (1 - u)^{-1}$;

(ii) P is a double point with tangent directions conjugate over GF(p); then define $L(E_p,u) := (1 + u)^{-1}$;

(iii) P is a cusp: then define $L(E_p,u) := 1$.

The <u>global zeta function</u> of E is obtained by taking the product of the local L-series over all primes p:

(1.4.2) $$\zeta_E(s) := \prod L(\tilde{E}_p,p^{-s}).$$

This converges for $\mathrm{Re}(s) > \frac{3}{2}$; it is conjectured that it has an analytic continuation to the entire s-plane and satisfies a functional equation similar to that of the Riemann zeta-function.

In order to write down the functional equation precisely, we must first define the <u>conductor</u> of E. This is

(1.4.3) $$N := \prod p^f$$

where the product is over all primes p, and f is determined as follows:

$f = 0$ if $p \nmid \Delta$;

$f = 1$ if $\tilde{E}_p$ has a double point;

$f \geq 2$ if $\tilde{E}_p$ has a cusp (if $p \nmid 2$ or 3 then $f = 2$ exactly).

There is an algorithm for determining f in any particular case, given in Tate's article in [2] .

Now if we let

$$\xi_E(s) := N^{\frac{1}{2}s}(2\pi)^{-s}\Gamma(s)\zeta_E(s),$$

where Γ is the Gamma function, then there is the following conjecture:

The function $\xi_E(s)$ is holomorphic in the entire s-plane and satisfies

(1.4.3) $$\xi_E(s) = w.\xi_E(2-s)$$

with $w = \pm 1$.

The sign of w has further significance which will emerge later.

The functional equation (1.4.3) has been proved, for example, in the following cases: when E has complex multiplication (then $\zeta_E(s)$ is a Hecke L-series with Grossencharacter: see Weil [26], Deuring [6]); and when E is a modular curve. In this latter case, $\zeta_E(s)$ is the Mellin transform of a modular form (c.f. Shimura [18]). Recall, for comparison, that the functional equation for the Riemann zeta-function has a proof which relies on the fact that its Mellin transform is the theta function $\theta(z) = \Sigma e^{2\pi i n^2 z}$ which is a modular form for the group generated by $z \to z+2$ and $z \to -1/z$.

§1.5 Connections

The conjecture referred to in the Introduction is that every elliptic curve defined over $\mathbb{Q}$ is in fact a modular curve. More precisely:

Conjecture *Let* E *be an elliptic curve defined over* $\mathbb{Q}$, *and* $\zeta_E(s) = \Sigma c_n n^{-s}$ *its zeta function. Then the function* $f(z) = \Sigma c_n e^{2\pi i n z}$, *for* z *in the upper half-plane, is a cusp form of weight* 2 *for* $\Gamma_0(N)$ *which is an eigenform for all the Hecke operators* T_p *for* $p \nmid N$, *and satisfies* $f|W = -wf$ *where* w *is the sign in the functional equation* (1.4.3). *Moreover, there is a rational map* $\varphi: X_0(N) \to E$, *defined over* $\mathbb{Q}$, *such that* $\omega \circ \varphi$ *is a multiple of the differential form* $f(z)dz$ *on* $X_0(N)$.

(From [21] p.197).

(Here N is the conductor of E, and ω is the standard differential on E: if E has equation (1.4.1) then $\omega = \dfrac{dx}{2y + a_1x + a_3}$).

In fact, the cusp forms arising from elliptic curves in this way should be precisely the newforms of §1.3. In particular, the number of newforms for $\Gamma_0(N)$ which are eigenforms of the Hecke algebra with rational eigenvalues should be equal to the number of isogeny classes of elliptic curves of conductor N defined over $\mathbb{Q}$. On the one hand, Tingley [23] computed the dimensions of S_N for $N < 330$ (and indeed, more than just the dimension); on the other hand, search programs for elliptic curves over $\mathbf{Q}$ with small conductor have been carried out. Agreement was found between the number of newforms for $\Gamma_0(N)$ with rational eigenvalues, and the number of isogeny classes of elliptic curves over $\mathbf{Q}$ with conductor N: see [2], introduction to Table 1.

The method Tingley used in [23] to calculate S_N for small values of N can be summarized as follows: by means of the duality between homology and cohomology on the Riemann surface $X_N(\mathbf{C})$, namely

$$< \gamma, \omega > \quad \rightarrow \quad \int_\gamma \omega$$

where $\gamma \in H_1(X_N(\mathbf{C}),\mathbf{C})$ and $\omega \in H^1(X_N(\mathbf{C}),\mathbf{C})$, there is an isomorphism between H^1 and H_1; one can calculate $H_1(X_N(\mathbf{C}),\mathbf{C})$ by means of modular symbols (this is explained in more detail in §4.1); one can define a Hecke action on homology itself which respects this isomorphism; this action can be computed explicitly in terms of modular symbols, leading to the splitting of the space into one-dimensional eigenspaces. To a rational one-dimensional eigenspace (that is, one with all eigenvalues rational), there then corresponds a cusp form of weight 2, whose coefficients are given in terms of the Hecke eigenvalues.

However one can, and Tingley did, go further than this: the algebraic curve $X_0(N)$ may be embedded in its Jacobean variety $J_0(N)$, and the one-dimensional rational eigenspaces correspond to elliptic curves defined over $\mathbf{Q}$ which are factors of $J_0(N)$; modular symbols give explicit cycles on $X_0(N)$, and by computing sufficiently many Hecke eigenvalues one may compute the corresponding forms, and hence their periods, to any desired degree of accuracy. Lastly, given the (approximate) periods of the elliptic curve, one can calculate an equation for the curve, with approximate coefficients. Tingley did this for each of the rational newforms he found, and came up with equations whose coefficients were very nearly integers; and in each case the corresponding equation with exact integral coefficients turned out to be a minimal equation of a curve with the correct conductor (namely the level N of the newform). In some cases, curves with certain conductors were found for the first time in this way.

§1.6 Complex Quadratic Fields

How much of the theory, conjecture, and computation described in §§1.1-1.5 can be generalized to a complex quadratic field in place of Q? This is the subject of the rest of this thesis. The most straightforward area to generalize is that of the elliptic curves; this will be done now.

Let K be a complex quadratic field with ring of integers O_K, and let E be an elliptic curve given by (1.4.1) with coefficients a_i in O_K. for each prime ideal p of O_K we can define a local L-series $L(\tilde{E}_p,u)$ just as in §1.4, where the reduced curve $\tilde{E}_p$ is now defined over O_K/p or GF(Np). The global zeta function is thus

$$\zeta_E(s) \quad := \quad \Pi\ L(\tilde{E}_p, N(p)^{-s})$$

with the product taken over all primes of O_K. The definition of the conductor as an ideal f of O_K is also carried out just as before. The conjectured functional equation now has the form

$$\xi_E(s) \quad = \quad \pm\ \xi_E(2-s)$$

where now $$\xi_E(s) \quad := \quad |D_K|^s (Nf)^{\frac{1}{2}s} (2\pi)^{-2s} (\Gamma(s))^2 \zeta_E(s).$$

(Here D_K is the absolute discriminant of K.) (See [16]).

Note that we can expand $\zeta_E(s)$ into a sum, just as in the rational case:

$$\zeta_E(s) \quad = \quad \sum_{\text{ideals } a} c(a)N(a)^{-s}$$

where each coefficient $c(a) \in Z$, and for primes p,

$$c(p) \quad = \quad 1 - M + N(p),$$

the 'Trace of Frobenius' at p,where M is the number of points on $\tilde{E}_p$, including the point at infinity.

The two-dimensional hyperbolic geometry of the upper half-plane H = SL(2,R)/SO(2,R) will be replaced by the three-dimensional hyperbolic geometry of 'upper half-space' H_3 = SL(2,C)/SU(2), on which $SL(2,O_K)$

acts discontinuously. This will be discussed in detail in the following Chapter.

As 'cusp forms of weight 2' for $SL(2,O_K)$ and its subgroups we introduce certain functions on H_3 which will correspond to invariant differentials (1-forms) on the three-dimensional quotient space. These will have Fourier expansions of a slightly more complicated form, but we will be able to define a Hecke action which will play the same rôle as in the rational theory. The notion of 'newform' will generalize, as will the connection between homology and cohomology: all this will be the subject of Chapter 3.

Explicit calculation of spaces of cusp forms is then possible by calculating the relevant homology and its Hecke action by means of suitably generalized modular symbols. Here we do not have the conjugation involution mentioned in §1.3, but instead we have an involution induced by the action of $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$, where ε generates the unit group O_K^* of O_K, which normalizes $\Gamma_0(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of O_K. This will play a similar, but not identical, rôle to conjugation in the rational theory. This is the subject of Chapter 4, where detailed algorithms will be given in the cases of the five Euclidean fields (where both the geometry and the algebra are simpler than in the general case), with remarks as to how they might be extended to the other fields with unique factorization as well as to the fields with class number greater than one.

In Chapter 5 we describe in detail the actual computations which have been carried out, and provide tables of the results, with some comments on them: in particular, these tables provide some evidence of the truth of a conjecture similar to the Weil conjecture above for elliptic curves over $\mathbb{Q}$.

Lastly, in Chapter 6, we show how a great deal of the theory in Atkin-Lehner [3] generalizes fairly easily, with modifications according to the local arithmetic, to apply to congruence subgroups of $SL(2,O_K)$ on the one hand, and the Fourier expansions of cusp forms of weight 2 for such subgroups on the other. In particular, we prove some results suggested by the patterns in the tables of Chapter 5.

CHAPTER 2

GL(2,C) and Hyperbolic Upper Half-space

In this Chapter, I will define the three-dimensional hyperbolic space, which will replace the upper half-plane as the space on which modular forms for complex quadratic fields are defined, derive the action of $GL(2,\mathbb{C})$ on it, and give some of its geometrical properties. This geometry will play an important part in Chapter 4, where it will be used in the proof of the algorithm for computing spaces of cusp forms by means of homology.

The main references here are Beardon's article [4] and Swan [20].

§2.1 Definition of H_3 and the action of $GL(2,\mathbb{C})$

Recall that every matrix in $SL(2,\mathbb{R})$ can be written in the form

$$(2.1.1)\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} y^{\frac{1}{2}} & y^{-\frac{1}{2}}x \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = y^{-\frac{1}{2}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} r(\theta)$$

where $r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Here x, y, and θ are given by the formulae

$$(2.1.2)\qquad x + iy = \frac{ai + b}{ci + d}; \qquad \theta = \arg(ci + d).$$

More generally, any matrix in $GL^+(2,\mathbb{R})$ may be written as

$$(2.1.3)\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} r(\theta)$$

with x, y and θ again given by (2.1.2). Thus we have a decomposition

$$GL^+(2,\mathbb{R}) = ZBK \quad\text{where}\quad Z = \{tI : t \in \mathbb{R}^*\},$$
$$B = \{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : y > 0,\ x,y \in \mathbb{R}\},$$
$$K = SO(2,\mathbb{R}).$$

So we can identify $PGL^+(2,\mathbb{R}) = PSL(2,\mathbb{R}) = BK$. Also since the subgroup B can be mapped bijectively to the upper half-plane H via $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \to x+yi$, we can express H as $PSL(2,\mathbb{R})/\bar{K}$, where $\bar{K} = PSO(2,\mathbb{R})$. Explicitly, we have a map

$$\pi : PSL(2,\mathbb{R}) \to H$$
$$(2.1.4)\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{ai + b}{ci + d}$$

and the inverse image of $i \in H$ is $\bar{K} = PSO(2,R)$. We can write (2.1.4) as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}(i)$ where on the right we have used the action defined in §1.1; it is then clear that, under the identification $B \leftrightarrow H$, the action of $PGL^+(2,R)$ defined in §1.1 is none other than the coset action on the coset space $PGL^+(2,R)/\bar{K}$.

Now there is a decomposition of $GL(2,C)$ similar to (2.1.3): every element of $GL(2,C)$ may be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \tag{2.1.5}$$

with $\alpha \neq 0$, $t \in R_+$, $z \in C$, and $|u|^2 + |v|^2 = 1$; here z and t are given by the formulae

$$z = (a\bar{c} + b\bar{d})(|c|^2 + |d|^2)^{-1}; \tag{2.1.6}$$

$$t = |ad - bc|(|c|^2 + |d|^2)^{-1}.$$

So we can write $GL(2,C) = ZBK$ where now $Z = \{\alpha I : \alpha \in C^*\}$;

$$B = \{\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} : t \in R,\ t > 0,\ z \in C\};$$

$$K = SU(2,C).$$

We may identify $PSL(2,C) = PGL(2,C) = BK$ and hence $B = PSL(2,C)/\bar{K}$, where $\bar{K} = PSU(2,C)$.

<u>Definition</u> *Upper Half-space is defined as*

$$H_3 := \{(z,t) : z \in C,\ t \in R,\ t > 0\}. \tag{2.1.7}$$

Clearly the map $\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \to (z,t)$ is a bijection from B to H_3; so we can identify $H_3 = PSL(2,C)/PSU(2)$. Now we can extract an action of $PSL(2,C)$ on H_3 from the natural coset action on B; direct calculation shows that this is given by the formulae

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z,t) \to (z',t') \quad \text{where} \tag{2.1.8}$$

$$z' = \frac{(az+b)\overline{(cz+d)} + (at)\overline{(ct)}}{|cz+d|^2 + |ct|^2} \quad ; \quad t' = \frac{|ad-bc|\,t}{|cz+d|^2 + |ct|^2}.$$

Denote the point $(0,1)$ in H_3 by j. It is a remarkable fact that if we identify the point $(z,t) \in H_3$ with the quaternion $h = z + tj$ we can express formulae (2.1.8) succinctly as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : h \to \frac{ah + b}{ch + d}, \tag{2.1.9}$$

where on the right-hand side we have quaternion division. In this notation we can express the projection

$$(2.1.10) \qquad \begin{array}{cccc} & PSL(2,C) & \to & PSL(2,C)/\bar{K} = B = H_3 \\ \text{as} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \to & \dfrac{aj+b}{cj+d} \end{array}$$

(compare (2.1.4)); the inverse image of j is just PSU(2).

§2.2 The Geometry of H_3

As a matter of notation and convention, we will identify C with $R^2 \times \{0\}$ in R^3, and thus H_3 with $\{(x,y,t) \in R^3 : t > 0\}$; denote the one-point compactifications of C and R^3 by C_∞ and R^3_∞ respectively; identify $i\infty \in C_\infty$ with $\infty \in R^3_\infty$; and identify C_∞ with the boundary of H_3 in R^3_∞. As coordinates for H_3 we will use either (z,t) or (x,y,t), so that $z = x + iy$, whichever is more convenient. The point at infinity will sometimes be referred to as $j\infty$.

An invariant metric for the action of $GL(2,C)$ on H_3 is given by $((dx)^2 + (dy)^2 + (dt)^2)/t^2$. The geometry is hyperbolic; geodesic lines are half-lines and semicircles perpendicular to the 'floor' C_∞; geodesic surfaces are half-planes and hemispheres perpendicular to the floor.

The action defined by (2.1.8) obviously makes sense when $t = 0$: hence we get an action of $GL(2,C)$ on the floor C_∞, which is identical to the usual action of $GL(2,C)$ on C_∞ by linear fractional transformations. Recall that a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,C)$ fixes one point of C_∞ if and only if $a+d = \pm 2$ (the point being ∞ in the case $c=0$), and otherwise two points. In the latter case, the fixed point set of α in H_3 is either the vertical semicircle joining the two fixed points, if they are both on the floor, or the vertical half-line joining the fixed point on the floor to $j\infty$.

As with the upper half-plane, we extend H_3 by including $j\infty$ and the points equivalent to it under the action of $GL(2,C)$, namely the points on the floor C, to form $H_3^* := H_3 \cup C \cup \{j\infty\}$.

The topology of H_3 is that induced by the invariant metric. This is in fact identical to the Euclidean topology of H_3 as a subset of R^3: c.f. Beardon [4]. We extend this topology to H_3^* as follows: a basis of open neighbourhoods for a point α in C is the set of all $S \cup \{\alpha\}$, where S is an open sphere whose boundary is tangent to C at α; a basis of open neighbourhoods at $j\infty$ is the collection of sets $\{(z,t) : t > t_0\} \cup \{j\infty\}$ for all $t_0 > 0$. The action of $GL(2,C)$ on H_3 is transitive: for the point j is carried to (z,t) by the matrix $\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}$; the stabilizer of j is the set of matrices of the form $\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$ with $u,v \in C$ and $|u|^2 + |v|^2 \neq 0$, since if $j = \dfrac{aj + b}{cj + d}$ (in the quaternion notation) then $aj + b = j(cj + d) = -\bar{c} + \bar{d}j$, and hence $a = \bar{d}$ and $b = -\bar{c}$.

§2.3 Discrete Subgroups and Fundamental Domains

As in the two-dimensional case, we say that a subgroup G of $SL(2,C)$ acts discontinuously on H_3 if every compact subset of H_3 meets only a finite number of its images under elements of G. As before, it is true that G acts discontinuously if and only if it is discrete (in the matrix topology): see Beardon (op.cit.) Theorem 4.2. For such a subgroup G, we define a fundamental domain or region for G to be a subset D of H_3 with the properties

(i) D is open in H_3;

(ii) Each orbit of G in H_3 meets D at most once, and meets the closure $\bar{D}$ of D at least once.

From (ii) it follows that $H_3 = \bigcup_{g \in G} g\bar{D}$, and that if $g \in G \setminus \{1\}$ then $gD \cap D = \phi$.

There is a general procedure for constructing fundamental domains, as follows. Select any point P_0 in H_3 not fixed by any element of $G \setminus \{1\}$; then the Dirichlet region D with centre P_0 is defined by

$$D := \{P \in H_3 : d(P,P_0) < d(P,gP_0) \quad \forall g \in G \setminus \{1\}\}$$

where $d(-,-)$ denotes the hyperbolic distance. Such regions have the following properties (c.f. Beardon op.cit.):

(i) D is a fundamental domain for G in H_3;

(ii) D is convex;

[Proof: let P_1, P_2 be distinct points in H_3 and define

$$S(P_1,P_2) := \{P \in H_3 : d(P,P_1) = d(P,P_2)\}$$

which is a hyperbolic plane such that P_1 and P_2 are inverse points with respect to the Euclidean sphere it determines; its complement in H_3 is the disjoint union of two convex half-spaces

$$H(P_1,P_2) := \{P \in H_3 : d(P,P_1) < d(P,P_2)\}$$

and $H(P_2,P_1)$. Then $D = \bigcap_{g\in G\setminus\{1\}} H(P_0,gP_0)$, an intersection of convex regions.]

(iii) D is locally finite: that is, each compact subset of H_3 meets only finitely many images under G of the closure $\overline{D}$;

(iv) The boundary of D is a countable number of geodesic line segments and polygons: a 'hyperbolic polyhedron';

(v) The map $\theta : G\backslash\overline{D} \to G\backslash H_3$, induced by the inclusion $\overline{D} \to H_3$, is a homeomorphism;

(vi) The faces of the polyhedron $\overline{D}$ are identified in pairs by certain elements of G: these elements generate G.

It follows from (v) that $G\backslash\overline{D}$ is topologically independent of the choice of point P_0 used to define D.

The groups G we will be interested in are subgroups of finite index in $PSL(2,O_K)$ where O_K is the ring of integers in the complex quadratic field K. In the next section we will give, as examples of the above situation, and for use in Chapter 4, fundamental regions for $PSL(2,O_K)$ for several fields K. These will all be Dirichlet regions as defined above.

§2.4 Fundamental regions for $SL(2,O_K)$

Let $K = \mathbb{Q}(\sqrt{-m})$, where m is a positive square-free integer, and let O_K be its ring of integers. Fix an integral basis $\{1,\alpha\}$ for K as follows: if $m \equiv 1,2 \pmod 4$ then $\alpha = \sqrt{-m}$; if $m \equiv 3 \pmod 4$ then $\alpha = \frac{1}{2}(1 + \sqrt{-m})$. In the cases m = 1,2 and 3 we will write "i", "θ", and "ρ" for "α" respectively.

We fix the following elements (*) of $PSL(2,O_K)$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; ; \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \; ; \quad U = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.$$

So the subgroup generated by S and T is the modular group $PSL(2,\mathbb{Z})$. These names will apply to all fields; other elements will be named for the particular cases as they are needed.

In [20], Swan finds fundamental regions for $PSL(2,O_K)$ for various complex quadratic fields K, and uses these to determine a presentation, in terms of generators and relations, for $PSL(2,O_K)$. Several of these fundamental regions were given by Bianchi [5] in the last century; however, Bianchi worked with the larger groups $PGL(2,O_K)$; especially in the case when the class number of K is greater than one, there are reasons for working with a larger group, under the action of which every cusp is equivalent to $j\infty$.

We now summarize the results for the five Euclidean fields (m = 1,2,3, 7 and 11). For these we can 'translate' any point (z,t) by suitable powers of T and U, which take (z,t) to (z+1,t) and (z+α,t) respectively, until $|z| < 1$: this is precisely because K is Euclidean. In fact, using S as well as T and U we can bring any point within the region

$$F \;:=\; \{(z,t) \,:\, z \in F_0,\; |z|^2 + t^2 \geq 1\}$$

where $$F_0 \;:=\; \{\, z \in \mathbb{C} \,:\, |z| \leq |z - z_0| \text{ for any } z_0 \in O_K\}.$$

(So F_0 is a rectangle when $m \equiv 1,2 \pmod 4$ and a hexagon when $m \equiv 3 \pmod 4$). Indeed, here is an algorithm for doing so:

[1] Apply suitable powers of T and U until $z \in F_0$;

(*) Here and throughout we will write elements of PGL and PSL as 2×2 matrices.

[2] If outside the unit sphere (i.e. if $|z|^2 + t^2 \geq 1$) then stop; else

[3] Apply S and go to [1].

The fact that this algorithm will stop follows from the observation that S multiplies the last coordinate of (z,t) by $(|z|^2 + t^2)^{-1}$, so that if (z,t) is inside the unit sphere then applying S 'raises' the point, i.e. increases its last coordinate; and the following lemma.

Lemma *For a fixed (z,t) there are only finitely many $t' > t$ such that (z,t) is equivalent under $PSL(2,O_K)$ to (z',t') for some z'.*

Proof: For fixed (z,t), and as c,d range over O_K, there are only finitely many values of $|cz + d|^2 + |c|^2t^2$ in the interval $(0,1)$.

Case $m = 1$ Here $PSL(2,Z[i])$ is generated by S,T,U and one further element $R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then $RS = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, which sends $(z,t) \to (-z,t)$. A fundamental region is obtained by cutting F in half:

$$D = \{(x + iy,t) : -\tfrac{1}{2} < x < \tfrac{1}{2},\ 0 < y < \tfrac{1}{2},\ x^2 + y^2 + t^2 > 1\}.$$

Also, we have

$$PSL(2,Z[i]) = \langle S,T,U,R \mid TU = UT,\ S^2=R^2=(RS)^2=(URS)^2=(TS)^3=(UR)^3=1\rangle .$$

Case $m = 2$ In this case F is itself a fundamental region for $PSL(2,Z[\theta])$ and no further generators are required:

$$PSL(2,Z[\theta]) = \langle S,T,U \mid TU=UT,\ S^2=(TS)^3=(SU^{-1}SU)^2=1 \rangle.$$

Case $m = 3$ Here we have an extra generator $R = \begin{pmatrix} 0 & \rho \\ \rho^2 & 0 \end{pmatrix}$, and $L := RS = \begin{pmatrix} \rho & 0 \\ 0 & \rho^2 \end{pmatrix}$ sends (z,t) to $(\rho^2 z,t)$. A fundamental region is obtained by cutting F in three:

$$D = \{(x + y\omega,t) : 0 < x,y < \tfrac{1}{2},\ |x + y\omega|^2 + t^2 > 1\},$$

where $\omega = \rho^2 = \frac{1}{2}(-1 + \sqrt{-3})$. A presentation for $PSL(2,Z[\rho])$ is

$$PSL(2,Z[\rho]) = \langle S,T,U,L \mid TU=UT,\ S^2=L^3=L^{-1}TLU=LTL^{-1}TU^{-1}=(TS)^3=(UT^{-1}SL)^3=1\rangle.$$

Case $m = 7$ Here F is itself a fundamental region for $PSL(2,Z[\alpha])$ and there are no extra generators:

$$PSL(2,Z[\alpha]) = \langle S,T,U \mid TU=UT,\ S^2=(TS)^3=(STU^{-1}SU)^2 = 1 \rangle.$$

Case $m = 11$ Again, F is itself a fundamental region and there are no extra generators:

$$PSL(2,Z[\alpha]) = \langle S,T,U \mid TU=UT,\ S^2=(TS)^3=(STU^{-1}SU)^3=1 \rangle.$$

§2.5 Congruence subgroups of $SL(2,0_K)$

We retain the notation of the previous section. Let

$$\Delta^{(K)} := SL(2,0_K).$$

Since in any particular context the field K will be fixed, we can usually omit the superscript and write Δ for $\Delta^{(K)}$.

For an ideal a of 0_K write N(a) for the ideal norm:

$$N(a) := [0_K:a] = \mathrm{Card}(0_K/a).$$

Define $$\varphi(a) := \mathrm{Card}((0_K/a)^\times),$$

the order of the multiplicative group modulo a. By analogy with the Euler φ function, we have the formula

$$\varphi(a) = N(a) \prod_{p|a}(1 - N(p)^{-1})$$

where the product is over all prime ideals p dividing a.

We define the <u>principal congruence subgroup of Δ of level a</u> as

$$\Delta(a) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta : a-1,\ b,\ c,\ d-1 \in a\};$$

that is, $\Delta(a)$ is the set of matrices in Δ 'congruent to the identity modulo a'.

From now on, in this section, suppose that K has class number one, so that 0_K is a principal ideal domain with unique factorization. Then the following sequence is exact:

$$\{I\} \longrightarrow \Delta(a) \longrightarrow \Delta \xrightarrow{\pi} SL(2,0_K/a) \longrightarrow \{I\}, \tag{2.5.1}$$

where π is induced by the natural projection $0_K \to 0_K/a$. The only part of this claim which is not obvious is the surjectivity of π: since 0_K is a principal ideal domain the proofs given by Shimura ([17] Lemma 1.38, p.20) or Ogg ([14], Proposition 13, Chapter IV) carry over, as indeed do the proofs of all the following formulae.

The index of $\Delta(a)$ in Δ is given by

$$[\Delta:\Delta(a)] = N(a)^3 \prod_{p|a} (1 - N(p)^{-2}).$$

We also define

$$\Delta_0(a) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta : c \in a\},$$

a subgroup of Δ which contains $\Delta(a)$ normally. The quotient group is isomorphic to

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in O_K/a, \quad ad \equiv 1 \pmod{a}\right\}$$

which clearly has order $N(a)\varphi(a)$; so we have

$$[\Delta_0(a):\Delta(a)] = \varphi(a)N(a) = N(a)^2 \prod_{p|a} (1 - N(p)^{-1})$$

and hence $$[\Delta:\Delta_0(a)] = N(a) \prod_{p|a} (1 + N(p)^{-1}).$$

The latter formula remains valid if we replace Δ and $\Delta_0(a)$ by the 'projectivizations' obtained by factoring out the scalar matrices in each to form $\overline{\Delta}$ and $\overline{\Delta}_0(a)$ respectively; this is because every scalar matrix in Δ obviously lies in $\Delta_0(a)$ for every ideal a of O_K.

CHAPTER 3

Cusp Forms of Weight 2 for Complex Quadratic Fields

In this Chapter I will discuss harmonic functions on hyperbolic three-space H_3 and define cusp forms of weight 2 for subgroups of $SL(2,O_K)$ of finite index. Here K is a complex quadratic field and O_K is its ring of integers. The harmonicity condition is the natural counterpart to the analyticity condition required of ordinary modular forms; the invariance condition for a function to be a 'form of weight 2' comes from the requirement that it should correspond to an invariant differential; the cuspidal condition again comes from a consideration of suitable Fourier expansions. I will also discuss the definition of Hecke operators on such functions , and show how they act on the Fourier expansions; define oldforms and newforms for $\Delta_0(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal of O_K; and show that one can calculate cusp forms of weight 2 by means of homology.

The theory developed in the first three sections is taken mainly from Weil's book [24]; however, Weil's approach is more general, in that he defines automorphic forms for a general global field, which are functions on GL(2) of the adèle group of the field. When the ground field is $\mathbb{Q}$, the general theory gives, as a special case, ordinary modular forms on the upper half-plane, from the single real embedding of $GL(2,\mathbb{Q})$ into $GL(2,\mathbb{R})$; similarly, in the case of a complex quadratic field which has a single pair of complex conjugate embeddings of $GL(2,K)$ into $GL(2,\mathbb{C})$, the general theory gives the automorphic forms on upper half-space which we describe here with no reference to adèles.

The results quoted in Section 3 are proved by Miyake in [13], which also uses the more general adèlic approach.

The relation between homology and cusp forms, discussed in Section 4, and the modular symbols defined there, have already been discussed and used by Kurčanov in [8].

§3.1 Harmonicity

Recall that we can express the upper half-plane as $H = GL(2,R)/Z.O(2,R)$ where Z is the group of scalar matrices. A complete set of coset representatives is given by the subgroup $B = \{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : y > 0,\ x \in R\}$. Write π for the projection $GL(2,R) \to H$; so the action of $GL(2,R)$ on $B = H$ is given by $g: b \to \pi(gb)$. The space H has the structure of a Riemannian symmetric space, with $ds^2 = ((dx)^2 + (dy)^2)/y^2$. A basis for the left-invariant differential forms on H is given by

$$(\beta_1, \beta_2) = (\frac{dz}{y}, -\frac{d\bar{z}}{y})$$

where $z = x + iy$. It is convenient to consider, as well as a differential form on H, its pullback to $G = GL(2,R)$. For $i = 1,2$ let ω_i be the differential form on G which coincides with $\pi^*\beta_i$ at the identity. A brief calculation shows that right translations by elements of $Z.O(2)$ operate on the ω_i by means of a 2-dimensional representation M of $Z.O(2)$ which is trivial on Z: if $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ then this representation is given by $\omega \to M(\kappa\zeta)^{-1}\omega$ where $\kappa \in O(2)$, $\zeta \in Z$, and M is defined as follows:

M is trivial on Z;

$$M\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{-2i\theta} \end{pmatrix} \quad ; \quad M\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

So a differential form on G is the inverse image of one on H if and only if it can be written as $\varphi_1\omega_1 + \varphi_2\omega_2$ where $\Phi = (\varphi_1, \varphi_2)$ is a vector-valued function on G satisfying $\Phi(g\kappa\zeta) = \Phi(g)M(\kappa\zeta)$ for all $g \in G$, $\kappa \in O(2)$, and $\zeta \in Z$.

In particular,

$$(3.1.1) \qquad \varphi_1(g.r(\theta)) = \varphi_1(g)e^{2i\theta} \quad \text{and} \quad \varphi_1(g.\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}) = \varphi_2(g).$$

In view of the second of these relations, not much information is lost by ignoring the 'antiholomorphic' component φ_2: in the classical theory, one just considers holomorphic differentials $f(z)dz$ instead of the more general $f_1(z)dz + f_2(z)d\overline{z}$; this is why in Chapter 1 we considered the action of $GL^+(2,R)$, the group of matrices with positive determinant, instead of $GL(2,R)$. No such simplification is possible in the complex case, as will be seen shortly.

Recall that on a Riemannian manifold V of dimension m there is a linear map from r-forms to $(m-r)$-forms called the adjoint or * operator. For a definition, and proofs of the following properties, c.f. de Rham [15]. If α and β are r-forms then

(i) $**\alpha = (-1)^{r(m+1)}\alpha$;

(ii) $\alpha \wedge *\beta = \beta \wedge *\alpha$;

(iii) $\alpha \wedge *\alpha = f\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_m$ where $f \geq 0$, and f is zero at exactly the points of V where α is zero;

(iv) $(\alpha,\beta) := \int_V \alpha \wedge *\beta$ is a scalar product.

Let d be the standard differentiation operator from r-forms to $(r+1)$-forms; then its transpose δ with respect to the above inner product is an operator of degree -1 (in fact $\delta = (-1)^r *^{-1} d\, *$). The * operator is its own transpose: $(*\alpha,*\beta) = (\alpha,\beta)$. Now let $\Delta = d\delta + \delta d$; this preserves degrees and is its own transpose; it commutes with d and δ; and $\Delta\alpha = 0$ if and only if $d\alpha = \delta\alpha = 0$, which is if and only if both α and $*\alpha$ are closed (by definition, a differential form β is <u>closed</u> if $d\beta = 0$).

A differential form α is said to be <u>harmonic</u> if $\Delta\alpha = 0$; equivalently, both α and $*\alpha$ must be closed.

For a 1-form $\omega = f_1(z)\beta_1 + f_2(z)\beta_2$ on the upper half-plane, we have $*\omega = -i(\overline{f}_1(z)\beta_2 - \overline{f}_2(z)\beta_1)$. This is closed if and only if $f_1\beta_1 - f_2\beta_2$ is closed, since $\overline{\beta}_1 = -\overline{\beta}_2$; so ω and $*\omega$ are both closed if and only if both $f_1\beta_1$ and $f_2\beta_2$ are closed; and this is if and only if both $f_1(z)\frac{dz}{y}$ and $\overline{f_2(z)\frac{dz}{y}}$ are holomorphic differentials in the upper half-plane.

Note that from (3.1.1) we have $f_1(\overline{z}) = f_2(z)$.

Next we will apply these ideas to determine the shape of harmonic differential forms on upper half-space H_3.

From (2.1.5) - (2.1.7) we have $H_3 = GL(2,\mathbf{C})/ZK$ where K is now $SU(2)$. Again a complete set of coset representatives is given by the subgroup $B = \{\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} : z \in \mathbf{C},\ t \in \mathbf{R},\ t > 0\}$. Write π for the projection from $G = GL(2,\mathbf{C})$ to H_3. So H_3 also has the structure of a Riemannian symmetric space, with $(ds)^2 = (dzd\bar{z} + (dt)^2)/t^2$. The space of left-invariant differential forms is now three-dimensional, with basis

$$\beta \quad = \quad (\beta_0, \beta_1, \beta_2) \quad = \quad (-\frac{dz}{t}, \frac{dt}{t}, \frac{d\bar{z}}{t}).$$

Again, denote the pullback of each β_i to G by ω_i. The effect of right translations defined by elements of KZ is now to operate on $\omega = (\omega_0\ \omega_1\ \omega_2)^t$ by a three-dimensional representation. This is clearly trivial on Z (which acts trivially on H_3). We now determine its action on K. Calculating the Jacobian matrix of the transformation (2.1.8) we find that

$$\frac{d(z',t',\bar{z}')}{d(z\ ,t\ ,\bar{z}\ )} = \frac{1}{(|r|^2 + |s|^2)^2} \begin{pmatrix} r^2\Delta & -2rs\Delta & s^2\Delta \\ r\bar{s}|\Delta| & (r\bar{r}-s\bar{s})|\Delta| & -\bar{r}s|\Delta| \\ \bar{s}^2\bar{\Delta} & 2\overline{rs\Delta} & \bar{r}^2\bar{\Delta} \end{pmatrix}$$

where $r = \overline{cz + d}$, $s = \bar{c}t$, and $\Delta = ad-bc$. In terms of the basis β for differentials, this becomes

$$(3.1.2) \qquad \beta' = |\Delta|^{-1}(|r|^2 + |s|^2)^{-1} \begin{pmatrix} r^2\Delta & -2rs\Delta & s^2\Delta \\ r\bar{s}|\Delta| & (r\bar{r}-s\bar{s})|\Delta| & -\bar{r}s|\Delta| \\ \bar{s}^2\bar{\Delta} & 2\overline{rs\Delta} & \bar{r}^2\bar{\Delta} \end{pmatrix} \beta$$

Now when $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$, say, where $|u|^2 + |v|^2 = 1$. At the identity in G we have $z = 0$ and $t = 1$. Substituting in (3.1.2) gives $\omega' = \rho(u,v)\omega$ where

$$\rho(u,v) \quad = \quad \begin{pmatrix} u^2 & 2uv & v^2 \\ -\bar{u}v & u\bar{u}-v\bar{v} & \bar{u}v \\ \bar{v}^2 & -2\overline{uv} & \bar{u}^2 \end{pmatrix} .$$

$-u\bar{v}$

Notice that if we write $\rho\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} = \rho(u,v)$ then $\rho: SU(2) \to SL(3,\mathbf{C})$ is the three-dimensional polynomial representation of $SU(2)$.

Hence a differential form on G is the inverse image of one on H_3 if and only if it can be written as $\Sigma\varphi_i\omega_i$ where $\Phi = (\varphi_0, \varphi_1, \varphi_2)$

satisfies $\Phi(g\kappa\zeta) = \Phi(g)\rho(\kappa\zeta)$ for all $g \in G$, $\kappa \in SU(2)$, $\zeta \in Z$. Let f_i be the function induced on $B = H_3$ by φ_i, for $i = 0, 1, 2$ and write $F = (f_0, f_1, f_2)$. As a particular case of the relation $F(g\kappa) = F(g)\rho(\kappa)$ let us record the following:

$$(3.1.3) \qquad F\left((z,t)\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}\right) = F(z,t) \cdot \begin{pmatrix} e^{2i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2i\theta}\end{pmatrix}.$$

Let $\omega = F.\beta = \sum_{i=0}^{2} f_i\beta_i$ be a 1-form on H_3. Then ω is harmonic if and only if ω and $*\omega$ are closed forms. From the definition of the $*$ operator one computes

$$(3.1.4) \qquad *(F.\beta) = -\tfrac{1}{2}i\overline{f}_1\,(\beta_0 \wedge \beta_2) + i\overline{f}_0\,(\beta_1 \wedge \beta_2) + i\overline{f}_2\,(\beta_1 \wedge \beta_2),$$

(c.f. [24] p.107).

<u>Definition</u> *A function* $F\colon H_3 \to \mathbb{C}^3$ *is said to be* <u>*harmonic*</u> *if*

(i) $F.\beta$ *is a harmonic differential form;*

(ii) F *is* <u>*slowly increasing*</u> *in the following sense: there exists* $N \geq 0$ *such that*

$$\left| F\left(\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}(z,t)\right)\right| = O(|x|^N)$$

as $x \to \infty$, $x \in \mathbb{R}$, *uniformly over compact sets in* H_3.

Here the modulus signs $|\ |$ on the left refer to any norm on $\mathbb{C}^3$.

Now suppose that $\Phi\colon G \to \mathbb{C}^3$ induces the function $F\colon H_3 \to \mathbb{C}^3$, that F is harmonic, and that Φ also satisfies the condition

$$(3.1.5) \qquad \Phi\left(\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}.g\right) = e^{-2\pi i(z+\overline{z})}\,\Phi(g) \qquad \text{for all } g \in GL(2).$$

Then F satisfies

$$F(z,t) = e^{-2\pi i(z+\overline{z})}F(0,t).$$

Writing $g_i(t) = f_i(0,t)$ we thus have $f_i(t) = e^{-2\pi i(z+\overline{z})}g_i(t)$. Then $F.\beta$ closed implies that $g_2(t) = -g_0(t)$ and $\frac{d}{dt}(t^{-1}g_0(t)) = 2\pi i t^{-1}g_1(t)$, while $*(F.\beta)$ closed implies from (3.1.4) that $\frac{d}{dt}(t^{-2}g_1(t)) = 4\pi i t^{-2}(g_2(t) - g_0(t))$. Let $s = 4\pi t$ and $K(s) = t^{-2}g_1(t)$. Then

$$sK''(s) + K'(s) - sK(s) = 0.$$

The only solution of this which does not increase exponentially as $s \to \infty$ is Hankel's function K_0; so we may take $g_1(t) = t^2K_0(4\pi t)$ and

$g_0(t) = -g_2(t) = -\frac{1}{2}it^2K_1(4\pi t)$, where Hankel's function K_1 is given by $K_1(s) = -\frac{d}{ds}(K_0(s))$.

We have proved

Proposition 3.1.6 *Let* $\Phi : G \to \mathbb{C}^3$ *satisfy* (3.1.5) *and induce* $F: H_3 \to \mathbb{C}^3$ *which is harmonic. Then*

$$F(z,t) = c\, e^{-2\pi i(z+\bar{z})}H(t)$$

where c *is a constant and*

(3.1.7) $$H(t) = (-\tfrac{1}{2}it^2K_1(4\pi t),\ t^2K_0(4\pi t),\ \tfrac{1}{2}it^2K_1(4\pi t)) .$$

§3.2 Fourier Expansions and the action of $GL(2,\mathbb{C})$ on Functions

Let $F:H_3 \to \mathbb{C}^3$ be a function which is harmonic, in the sense of the previous section. We will now define an action of $GL(2,\mathbb{C})$ on such functions which will generalize Definition (1.3.1). For $g \in GL(2,\mathbb{C})$, we wish to define a function $F|g$ such that $F.\beta$ is an invariant differential under the action of a discrete subgroup G of $GL(2,\mathbb{C})$ if and only if $F|g = F$ for all $g \in G$. Rewrite (3.1.2) in the form

(3.2.1) $$\beta' = J(g;(z,t))\beta$$

where if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we set $\Delta = ad-bc$, $r = \overline{cz+d}$, $s = \bar{c}t$ and then

(3.2.2) $$J(g;(z,t)) = \frac{1}{|\Delta|\,(|r|^2+|s|^2)}\begin{pmatrix} r^2\Delta & -2rs\Delta & s^2\Delta \\ r\bar{s}|\Delta| & (r\bar{r}-s\bar{s})|\Delta| & -\bar{r}s|\Delta| \\ \bar{s}^2\bar{\Delta} & 2\overline{rs\Delta} & \bar{r}^2\bar{\Delta} \end{pmatrix}.$$

Definition(*) *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C})$ *and* $F:H_3 \to \mathbb{C}^3$ *be as above. Then define a new function* $F|g$ *by*

(3.2.3) $$(F|g)(z,t) = F(g(z,t))\,J(g;(z,t)).$$

(*) This definition is not quite analogous to definition (1.3.1) in the rational case, since there the definition is designed to make f(z)dz, and not f(z)dz/y, invariant although dz/y is the invariant measure on the upper half-plane. The course we have adopted seems most natural here; the formulae in this Chapter would need modification if the alternative were used.

Note that $(F|g)(z,t).\beta(z,t) = F(g(z,t))J(g;(z,t)).\beta(z,t) = F(g(z,t)).\beta(g(z,t))$ by (3.2.3) and (3.2.1); so that the differential $F.\beta$ is invariant under g if and only if $F|g = F$.

As it will sometimes be convenient in the sequel to pass between functions F on H_3 and functions Φ on $GL(2,C)$, we give the appropriate formulae here. Given $\Phi : GL(2,C) \to C^3$ we let F be the restriction of Φ to B, which as usual we identify with H_3. On the other hand, given $F : H_3 \to C^3$ we define for $g \in GL(2,C)$,

$$\Phi(g) := F(g(j))J(g;j)$$

where j, as before, denotes the point (0,1) in H_3. Note that in fact $\Phi(g) = (F|g)(j)$, from (3.2.3), and that for $g \in GL(2,C)$, $\kappa \in SU(2)$ and $\zeta \in Z$ we have

$$\begin{aligned}\Phi(g\kappa\zeta) &= F(g\kappa\zeta(j))J(g\kappa\zeta;j)\\ &= F(g(j))J(g;j)J(\kappa\zeta;j)\\ &= \Phi(g)\ \rho(\kappa\zeta)\end{aligned}$$

as desired, since when the Jacobian function J is restricted to KZ and evaluated at j it coincides with the representation ρ (by definition of ρ). Here we have used the fact that J satisfies the 'chain rule' or cocycle identity

$$J(g_1g_2;(z,t)) = J(g_1;g_2(z,t))J(g_2;(z,t)).$$

It will be useful later to have an explicit formula for the special case of (3.2.3) when $g = \begin{pmatrix} a & b \\ 0 & d\end{pmatrix}$. From (2.1.8) we have $\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}(z,t) = \left(\frac{az+b}{d}, \left|\frac{a}{d}\right|t\right)$. In (3.2.2) we thus have $r=\bar{d}$, $s=0$, $\Delta = ad$, and so $J\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix};(z,t)\right) = \mathrm{diag}\left(\frac{a\bar{d}}{|ad|},1,\frac{\bar{a}d}{|ad|}\right) = \mathrm{diag}(e^{i\theta},1,e^{-i\theta})$ where $\theta = \arg(a/d)$.
Now also from (3.1.3) we can write

$$F\left(\frac{az+b}{d}, \left|\frac{a}{d}\right|t\right) = F\left(\frac{az+b}{d},\frac{a}{d}t\right)\mathrm{diag}(e^{-i\theta},1,e^{i\theta})$$

so that the diagonal matrices cancel out, giving us

(3.2.4) $$\left(F\left|\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\right.\right)(z,t) = F\left(\frac{az+b}{d}, \frac{a}{d}t\right).$$

Note that on the right-hand side of (3.2.4) the second argument is not necessarily positive real; it is frequently the case that simpler formulae are obtained by writing them this way: we can convert from one form to the other by means of the equation

$$(3.2.5) \qquad F(z,te^{i\theta}) = F(z,t)\mathrm{diag}(e^{i\theta}, 1, e^{-i\theta})$$

which follows from (3.1.3) since

$$\begin{pmatrix} te^{i\theta} & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\frac{1}{2}i\theta} & 0 \\ 0 & e^{\frac{1}{2}i\theta} \end{pmatrix} \begin{pmatrix} z & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}i\theta} & 0 \\ 0 & e^{-\frac{1}{2}i\theta} \end{pmatrix} .$$

This difficulty was avoided in the real case by restricting to matrices of $GL(2,R)$ with positive determinant: in the present case it is more convenient just to allow arbitrary non-zero complex numbers as the second argument of functions. Indeed, there is a case for replacing the upper half-space H_3 with $\{(z,w) \in C^2 : w \neq 0\}$; then one could identify (z,w) with the quaternion $z + wj$ and gain much simplification in formulae (compare (2.1.8) with (2.1.9)). There might also then be a possibility of a complex structure. However this line will not be pursued here.

For 'translation' matrices $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ which send (z,t) to $(z+b,t)$, formula (3.2.4) simplifies even further to $(F \mid \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})(z,t) = F(z+b,t)$ since the Jacobian matrix is then trivial.

Now let K be a complex quadratic field, and O_K its ring of integers. Let Γ be a discrete subgroup of $GL(2,K)$ containing all the translations $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ for $\alpha \in O_K$: for example Γ could be the congruence subgroup $\Delta_0(a)$ defined in §2.5. Then if our harmonic function F is invariant under Γ, we have in particular

$$(3.2.6) \qquad F(z + \alpha, t) = F(z,t)$$

for all $\alpha \in O_K$. Fix t and consider F as a function of z alone: it then follows that F has a Fourier expansion with respect to the characters of C^+ (the additive group) which are trivial on O_K. What do these characters look like? If ψ is any non-trivial character of C^+, then for any fixed $w \in C$ the function $z \to \psi(wz)$ is also a character, and in fact all characters of C^+ have this form (c.f. Tate's thesis [22] §2.2), so that C^+ may be identified with its character group. To fix this identification we will use a particular character ψ, namely

(3.2.7) $$\psi(z) := e^{-2\pi i(z+\bar{z})}.$$

So F has an expansion of the form

$$F(z,t) = c_0(t) + \sum_{\alpha} c(\alpha,t)\psi(\alpha z).$$

As we know that $F(z+\beta,t) = F(z,t)$ for all $\beta \in O_K$,

$$\Sigma\, c(\alpha,t)\psi(\alpha z) = \Sigma\, c(\alpha,t)\psi(\alpha(z+\beta)) = \Sigma\, c(\alpha,t)\psi(\alpha\beta)\psi(\alpha z)$$

and hence for all α, and all $\beta \in O_K$,

$$c(\alpha,t) = c(\alpha,t)\psi(\alpha\beta).$$

This means that $c(\alpha,t) = 0$ unless α is such that $\psi(\alpha\beta) = 1$ for all $\beta \in O_K$. But $\psi(\alpha\beta) = \exp(-2\pi i \mathrm{Tr}(\alpha\beta)) = 1$ if and only if $\mathrm{Tr}(\alpha\beta) \in Z$, so $\beta \to \psi(\alpha\beta)$ is the trivial map if and only if $\mathrm{Tr}(\alpha\beta) \in Z$ for all $\beta \in O_K$, which is if and only if α belongs to the 'inverse different' δ^{-1} of K. So the Fourier expansion takes the form

(3.2.8) $$F(z,t) = c_0(t) + \sum_{\alpha\in\delta^{-1}} c(\alpha,t)\psi(\alpha z).$$

The different δ is an ideal of O_K: in fact it is the principal ideal $(\sqrt{D})O_K$ where D is the discriminant of K. Writing $\eta = \sqrt{D}$ and substituting α for $\eta\alpha$ gives

(3.2.9) $$F(z,t) = c_0(t) + \sum_{\alpha\in O_K} c(\eta^{-1}\alpha,t)\psi(\eta^{-1}\alpha z).$$

The coefficient functions $c_0(t)$ and $c(\alpha,t)$ are given by the usual formulae:

(3.2.10) $$\begin{aligned} c_0(t) &= \int_{O_K\backslash C} F(z,t)dz; \\ c(\alpha,t) &= \int_{O_K\backslash C} \psi(-\alpha w)F(w,t)dw. \end{aligned}$$

Writing the latter as

$$c(\alpha,t) = \int \psi(-\alpha w)F\left(\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}(0,t)\right)dw = \int \psi(-\alpha w)\ \Phi\left(\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\right)dw,$$

where Φ is the associated function on GL(2), leads us to define more generally for $g \in GL(2)$:

(3.2.11) $$c(\alpha;g) = \int_{O_K\backslash C} \psi(-\alpha w)\ \Phi\left(\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}g\right)dw.$$

Since F was assumed to be harmonic, the same is true of $c(\alpha;g)$, as a function of g for fixed α. From (3.2.11) we have

$$\begin{aligned} c\left(\alpha;\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}g\right) &= \int_{O_K\backslash C} \psi(-\alpha w)\ \Phi\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}g\right)dw \\ &= \int_{O_K\backslash C} \psi(-\alpha w)\ \Phi\left(\begin{pmatrix}1 & z+w\\ 0 & 1\end{pmatrix}g\right)dw \end{aligned}$$

$$= \int \psi(-\alpha(w-z))\, \Phi\left(\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix} g\right) dw$$
$$= \psi(\alpha z) c(\alpha;g).$$

We cannot apply Proposition 3.1.6 directly to this since we have the character $\psi(\alpha z)$ instead of $\psi(z)$. But if we define $c'(\alpha;g) := c(\alpha; \begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix} g)$ then c' satisfies

$$c'(\alpha; \begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix} g) = \psi(z) c'(\alpha;g).$$

Hence by Proposition 3.1.6 there is a constant $c_1(\alpha)$ such that

$$c'(\alpha; \begin{pmatrix}t & z\\ 0 & 1\end{pmatrix}) = c_1(\alpha)\psi(z)H(t)$$

where $H(t)$ is given in (3.1.7). Substituting for c gives $c(\alpha; \begin{pmatrix}t & z\\ 0 & 1\end{pmatrix}) = c_1(\alpha)\psi(\alpha z)H(\alpha t)$ and hence $c(\alpha,t) = c_1(\alpha)H(\alpha t)$. Writing $c(\alpha)$ for $c_1(\eta^{-1}\alpha)$ gives us the following result.

Proposition 3.2.12 *Let $F : H_3 \to \mathbf{C}^3$ be a harmonic function invariant under all $\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}$ for $\alpha \in \mathbf{O}_K$. Then F has a Fourier expansion of the form*

(3.2.13) $$F(z,t) = c_0(t) + \sum_{\alpha \in \mathbf{O}_K} c(\alpha)H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha z)$$

where: $c_0(t)$ *is given by* (3.2.10);

$c(\alpha)$ *is a coefficient depending on* α;

$H(t)$ *is given by* (3.1.7)

and ψ is the standard character (3.2.7).

Fourier expansions of the form (3.2.13) are special in two ways. First of all, we assumed that F was invariant under $\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}$ for <u>every</u> $\alpha \in \mathbf{O}_K$: that is, that the cusp at $j\infty$ was of 'width 1'. More generally, if the set of α such that F is invariant under $\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}$ form the ideal h of $\mathbf{O}_K$, we must replace the inverse different δ^{-1} by the $\mathbf{O}_K$-module dual to h: namely $\{\alpha \in K ; Tr(\alpha\beta) \in \mathbf{Z} \text{ for all } \beta \in h\}$ which is just $h^{-1}\delta^{-1}$. If we assume that $\mathbf{O}_K$ is a principal ideal domain, then h is principal, generated by an element λ say; and the Fourier series for F takes the form

(3.2.14) $$F(z,t) = c_0(t) + \sum_{\alpha \in \mathbf{O}_K} c(\alpha)H(\lambda^{-1}\eta^{-1}\alpha t)\psi(\lambda^{-1}\eta^{-1}\alpha z)$$

where now

(3.2.15) $$c_0(t) = \int_{\mathbf{O}_K \backslash \mathbf{C}} F(\lambda z,t)\, dz.$$

Also, we may consider Fourier expansions at other cusps. All the

'K-rational' cusps $(s,0)$ for $s \in K$ are equivalent to $j\infty$ under the action of $SL(2,\mathcal{O}_K)$, provided that K has class number one. More generally, the number of orbits of $SL(2,\mathcal{O}_K)$ acting on $P_1(K) = K \cup \{j\infty\}$ is equal to the class number of K. If $\sigma \in SL(2,\mathcal{O}_K)$ sends $j\infty$ to the cusp s, and F is invariant under a subgroup Γ, then $F|\sigma$ is invariant under all matrices of the form $\rho = \sigma^{-1}\gamma\sigma$ for $\gamma \in \Gamma_s$, where

$$\Gamma_s = \{ \gamma \in \Gamma \ : \gamma s = s\} ,$$

since
$$\begin{aligned}((F|\sigma)|\rho)(P) &= (F|\sigma\rho)(P) = (F|\gamma\sigma)(P) \\ &= F(\gamma\sigma(P))J(\gamma\sigma;P) \\ &= F(\gamma(\sigma(P)))J(\gamma;\sigma(P))J(\sigma;P) && \text{(cocycle identity)} \\ &= (F|\gamma)(\sigma(P))J(\sigma;P) \\ &= F(\sigma(P))J(\sigma;P) && \text{(since } \gamma \in \Gamma) \\ &= (F|\sigma)(P)\end{aligned}$$

for $P = (z,t) \in H_3$. But $\sigma^{-1}\Gamma_s\sigma$ fixes $j\infty$ and so consists of matrices of the form $\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}$: if the set of α such that $F|\sigma$ is invariant under $\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}$ is the ideal $h = (\lambda)$, then as before $F|\sigma$ has a Fourier expansion of the form (4.2.14). We now define this to be the <u>Fourier expansion of F at the cusp s.</u> Its 'zeroth' coefficient is given by

(3.2.16) $$\int_{\mathcal{O}_K\backslash C} (F|\sigma)(\lambda z,t)\ dt$$

This leads us to define automorphic forms and cusp forms as follows.

<u>Definition 3.2.17</u> *Let K be a complex quadratic field with class number one and ring of integers $\mathcal{O}_K$; let Γ be a subgroup of $SL(2,\mathcal{O}_K)$ of finite index. Then an <u>automorphic form of weight 2 for Γ</u> is a function $F : H_3 \to C^3$ satisfying*

(i) F is harmonic;

(ii) $F|\gamma = F$ for all $\gamma \in \Gamma$.

If, in addition, for all $\sigma \in SL(2,\mathcal{O}_K)$ and all $t \geq 0$, F satisfies

(iii) $$\int_{\mathcal{O}_K\backslash C} (F|\sigma)(\lambda z,t)\ dz = 0$$

where (λ) is the width of the cusp at $\sigma(j\infty)$ as defined above, then F is called a <u>cusp form of weight 2 for Γ</u>.

Notes: (i) In this thesis only forms of weight 2 will be considered, so from now on the qualification 'of weight 2' will be omitted.

(ii) Cusp forms F for Γ correspond to harmonic differentials $F.\beta$ on the quotient space which are zero at the cusps of $\Gamma\backslash H_3^*$.

(iii) Note that in the Fourier expansion (3.2.13) the argument of H is $\eta^{-1}\alpha t$ where $\eta, \alpha \in O_K$ and $t \in R$, $t > 0$, whereas in (3.1.7) we defined H only for positive real arguments. This abuse of notation is resolved by (3.2.5) and the remarks following it: we set

$$\begin{aligned} H(re^{i\theta}) &= H(r)\,\mathrm{diag}(e^{i\theta},1,e^{-i\theta}) \\ &= (-\tfrac{1}{2}ir^2e^{i\theta}K_1(4\pi r),\ r^2K_0(4\pi r),\ \tfrac{1}{2}ir^2e^{-i\theta}K_1(4\pi r)). \end{aligned}$$

(iv) The only part of the definition which depends on Γ itself is (ii), the invariance condition. So if F is an automorphic form or cusp form for Γ, and F is also invariant under another group Γ', then F is also a form for Γ'.

§3.3 Hecke operators

As with ordinary modular forms, we can extend the action of $GL(2,\mathbb{C})$ on functions, given by (3.2.3), to the group ring of $GL(2,\mathbb{C})$; Hecke operators will be defined by particular elements of this group ring. Before we define them, however, we need to introduce some other operators.

Let ε be a unit of O_K and let I_ε denote the matrix $\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}$. The action of I_ε on H_3 is to send (z,t) to $(\varepsilon z,t)$ since $|\varepsilon| = 1$. Also, $J(I_\varepsilon;(z,t)) = \mathrm{diag}(\varepsilon,1,\varepsilon^{-1})$ and from (3.2.4),

$$(3.3.1)\qquad (F|I_\varepsilon)(z,t) = F(\varepsilon z,\varepsilon t) = F(\varepsilon z,t)\mathrm{diag}(\varepsilon,1,\varepsilon^{-1}).$$

We can also compute the action of I_ε on Fourier coefficients: if F has a Fourier expansion given by (3.2.13) then

$$\begin{aligned}(F|I_\varepsilon)(z,t) &= F(\varepsilon z,\varepsilon t)\\ &= c_0(\varepsilon t)+\sum_{\alpha\in O_K} c(\alpha)H(\eta^{-1}\alpha\varepsilon t)\psi(\eta^{-1}\alpha\varepsilon z)\\ &= c_0(\varepsilon t)+\sum_{\alpha\in O_K} c(\alpha\varepsilon^{-1})H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha z).\end{aligned}$$

So if F is invariant under I_ε we must in fact have

$$(3.3.2)\qquad c(\alpha) = c(\alpha\varepsilon)$$

for all $\alpha \in O_K$.

Denote by $S(\mathfrak{a})$ the space of all cusp forms for $\Delta_0(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal of O_K, and suppose $F \in S(\mathfrak{a})$. Since $\begin{pmatrix}\varepsilon^2 & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}$ are projectively equivalent they give the same transformation of H_3; but the latter is in $\Delta_0(\mathfrak{a})$ for every $\mathfrak{a}$; so we must have $F|I_{\varepsilon^2} = F$. Then by (3.3.2), the coefficients of F satisfy

$$(3.3.3)\qquad c(\alpha) = c(\varepsilon^2\alpha)$$

for every unit $\varepsilon \in O_K^*$. Also note that I_ε normalizes $\Delta_0(\mathfrak{a})$ so that $F|I_\varepsilon$ is invariant under $\Delta_0(\mathfrak{a})$ if and only if F is. So if ε_0 is a generator of the unit group O_K^* then I_{ε_0} induces an involution of $S(\mathfrak{a})$ which we denote by J and call the 'main involution' of $S(\mathfrak{a})$. Hence we can split $S(\mathfrak{a})$ up as

$$(3.3.4)\qquad S(\mathfrak{a}) = S^+(\mathfrak{a}) \oplus S^-(\mathfrak{a})$$

where J acts as $+1$ on $S^+(\mathfrak{a})$ and as -1 on $S^-(\mathfrak{a})$. Note that by (3.3.2), for $F \in S^+(\mathfrak{a})$, the Fourier Coefficients $c(\alpha)$ satisfy $c(\alpha) = c(\varepsilon_0\alpha)$, and so depend only on the ideal (α); whereas for $F \in S^-(\mathfrak{a})$ we have $c(\alpha) = -c(\varepsilon_0\alpha)$ and this is not so.

Now suppose that K has class number one so that O_K is a principal ideal domain. Let $\mathfrak{p}$ be a prime ideal of O_K generated by an element π. Then we define the Hecke operator T_π to be the transformation $F \to F|T_\pi$, where

(3.3.5) $$T_\pi = \sum_{\alpha \bmod \pi} \begin{pmatrix} 1 & \alpha \\ 0 & \pi \end{pmatrix} + \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}.$$

For a unit $\varepsilon \in O_K$ we have

$$T_{\varepsilon\pi} = \sum_{\alpha \bmod \pi} \begin{pmatrix} 1 & \alpha \\ 0 & \varepsilon\pi \end{pmatrix} + \begin{pmatrix} \varepsilon\pi & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \sum \begin{pmatrix} 1 & \alpha \\ 0 & \pi \end{pmatrix} + \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{so that}$$

(3.3.6) $$T_{\varepsilon_0\pi} = JT_\pi .$$

Hence we cannot in general define an operator $T_\mathfrak{p}$ for a prime ideal $\mathfrak{p}$, since the definition may depend on the choice of generator for $\mathfrak{p}$.

<u>Lemma 3.3.7</u> *If F is in $S(\mathfrak{a})$ then so is $F|T_\pi$ for every prime element $\pi \nmid \mathfrak{a}$.*

Proof: Condition (i) of Definition 3.2.17 is automatically satisfied by $F|L$, for every element L of the group ring of $GL(2,\mathbb{C})$, if it is satisfied by F. As for the cuspidal condition (iii): this is also satisfied by $F|L$, for L in the group ring of $GL(2,K)$, since F itself vanishes at each K-rational cusp. So in this and similar results, it suffices to show that $F|L$ is invariant under the group concerned.

In this case, the proof of the fact that $F|T_\pi$ is invariant under $\Delta_0(\mathfrak{a})$ provided that $\pi \nmid \mathfrak{a}$ is almost identical to the proof of the corresponding fact for $\Gamma_0(N)$ in the rational case (c.f. Atkin-Lehner [3] Lemma 6). The modifications are trivial and so the proof will not be written out here.

Suppose F has a Fourier expansion of the form (3.2.13):

$$F(z,t) = c_0(t) + \sum_{\alpha \in O_K} c(\alpha) H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha z).$$

Then $$(F|T_\pi)(z,t) = \sum_{\beta \bmod \pi} \left(F\left|\begin{pmatrix} 1 & \beta \\ 0 & \pi \end{pmatrix}\right.\right)(z,t) + \left(F\left|\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}\right.\right)(z,t).$$

Now from (3.2.4),

$$\left(F\left|\begin{pmatrix}1 & \beta\\ 0 & \pi\end{pmatrix}\right.\right)(z,t) = F\left(\frac{z+\beta}{\pi}, \frac{t}{\pi}\right)$$

and
$$\left(F\left|\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}\right.\right)(z,t) = F(\pi z, \pi t).$$

Hence
$$(F|T_\pi)(z,t) = \sum_{\beta \bmod \pi} F\left(\frac{z+\beta}{\pi}, \frac{t}{\pi}\right) + F(\pi z, \pi t).$$

Substituting in the Fourier expansion (3.2.13) gives

$$\begin{aligned}(F|T_\pi)(z,t) &= N(\pi)c_0\left(\frac{t}{\pi}\right) + c_0(\pi t)\\ &\quad + \sum_{\beta \bmod \pi}\ \sum_{\alpha} c(\alpha)H\left(\eta^{-1}\alpha\frac{t}{\pi}\right)\psi\left(\eta^{-1}\alpha\frac{z+\beta}{\pi}\right)\\ &\quad + \sum_{\alpha} c(\alpha)H(\eta^{-1}\alpha\pi t)\psi(\eta^{-1}\alpha\pi z)\\ &= c_0'(t) + \sum_{\alpha} c'(\alpha)H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha z), \text{ say, where}\end{aligned}$$

$$(*)\qquad \begin{aligned} c_0'(t) &= N(\pi)c_0(t\pi^{-1}) + c_0(t\pi), \text{ and}\\ c'(\alpha) &= N(\pi)c(\alpha\pi) + c(\alpha/\pi).\end{aligned}$$

(We use the convention that $c(\alpha) = 0$ unless $\alpha \in 0_K$.)

Here we have used the fact that

$$\sum_{\beta \bmod \pi} \psi(\eta^{-1}\alpha\beta/\pi) = \begin{cases} N(\pi) & \text{if } \pi \mid \alpha;\\ 0 & \text{if } \pi \nmid \alpha.\end{cases}$$

From (*) it follows that if F is a cusp form then so is $F|T_\pi$. Suppose also that F is an 'eigenform' for T_π, so that $F|T_\pi = \lambda_\pi F$ for some constant λ_π. Then $c'(\alpha) = \lambda_\pi c(\alpha)$ for all α. Write $a(\alpha) = N(\alpha)c(\alpha)$ and $a'(\alpha) = N(\alpha)c'(\alpha)$, so that (*) becomes

(3.3.8) $$a'(\alpha) = a(\alpha\pi) + N(\pi)a(\alpha/\pi).$$

Then substituting $\alpha = 1$ gives

(3.3.9)(i) $$\lambda_\pi a(1) = a(\pi)$$

and for $r \geq 1$, $$a(\pi^{r+1}) = \lambda_\pi\, a(\pi^r) - N(\pi)a(\pi^{r-1}).$$

Also if $\pi \nmid \alpha$, (3.3.8) gives $a(\alpha\pi) = \lambda_\pi\, a(\alpha)$, and for $r \geq 1$,

(3.3.9)(ii) $$a(\alpha\pi^{r+1}) = \lambda_\pi\, a(\alpha\pi^r) - N(\pi)a(\alpha\pi^{r-1}).$$

In particular, if F is an eigenform for all the T_π, the coefficients of F may all be computed in terms of the eigenvalues λ_π. Moreover from (3.3.9) it follows by simple induction that

$$a(1)a(\alpha\pi^r) = a(\alpha)a(\pi^r) \qquad \text{for } \pi \nmid \alpha$$

so that the coefficients become multiplicative if we normalize by setting $a(1) = 1$. (We can do this because $a(1) \neq 0$ for an eigenform: this fact is not trivial to prove — see the end of this section.)

For each prime π dividing the level $\mathfrak{a}$ there is an involution W_π, defined just as in the rational case: let π^r be the highest power of π dividing $\mathfrak{a}$ and let W_π be any matrix

$$\begin{pmatrix} \pi^r x & y \\ \alpha z & \pi^r w \end{pmatrix} \qquad \text{with determinant } \pi^r$$

where $\mathfrak{a} = (\alpha)$; then if F is a cusp form for $\Delta_0(\mathfrak{a})$, so is $F|W_\pi$, which is independent of the choice of matrix chosen, and $(F|W_\pi)|W_\pi = F$.

We finish this section with a summary of some results concerning the action of the Hecke operators on the space of cusp forms $S(\mathfrak{a})$, for $\mathfrak{a}$ an ideal of 0_K. They are all special cases of results valid for automorphic forms over an arbitrary global field, which can be found with proofs in Miyake [13], and are straightforward generalizations of the corresponding facts about cusp forms in the rational case. The proofs are very similar to those in Atkin-Lehner [3] and could have been written out in full here in the present context of a complex quadratic field, but this seemed unnecessary.

<u>Summary of results about Cusp Forms and Hecke Operators</u>

If $(\pi_1) \neq (\pi_2)$ then $T_{\pi_1} T_{\pi_2} = T_{\pi_2} T_{\pi_1}$.

If $(\pi_1) \neq (\pi_2)$ and $\pi_1, \pi_2 \mid \mathfrak{a}$ then $W_{\pi_1} W_{\pi_2} = W_{\pi_2} W_{\pi_1}$.

If $\pi_1 \mid \mathfrak{a}$ and $\pi_2 \nmid \mathfrak{a}$ then $T_{\pi_2} W_{\pi_1} = W_{\pi_1} T_{\pi_2}$.

There is an inner product on $S(\mathfrak{a})$ such that each T_{π_1} $(\pi_1 \nmid \mathfrak{a})$ and each W_{π_2} $(\pi_2 \mid \mathfrak{a})$ is self-adjoint with respect to it. Hence there is a basis of $S(\mathfrak{a})$ consisting of forms which are eigenvectors, or <u>eigenforms</u>, for all the T_π for $\pi \nmid \mathfrak{a}$ and all the W_π for $\pi \mid \mathfrak{a}$. Elements of such a basis always have their first coefficient $a(1) \neq 0$; normalize them so that $a(1) = 1$.

If $F \in S(\mathfrak{b})$ where $\mathfrak{b} \mid \mathfrak{a}$ then $F\left|\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right. \in S(\mathfrak{a})$ for any $k \mid \mathfrak{a}\mathfrak{b}^{-1}$; denote by $S^{old}(\mathfrak{a})$ the subspace generated by all such $F\left|\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right.$ for all

$b \mid a$; then $S^{old}(a)$ is mapped to itself by all the T_π for $\pi \nmid a$ and W_π for $\pi \mid a$, and we may form the orthogonal complement $S^{new}(a)$ of $S^{old}(a)$ with respect to the above inner product. This space $S^{new}(a)$ is spanned by the eigenforms, called newforms, which are not oldforms (elements of $S^{old}(a)$). So $S^{new}(a)$ has a basis consisting of newforms, which are eigenforms for all the T_π for $\pi \nmid a$ and the W_π for $\pi \mid a$, and have first coefficient 1.

The algebra generated by all the T_π, restricted to $S^{new}(a)$, is commutative, semisimple, and has rank equal to $\dim(S^{new}(a))$.

The following result is deeper, and correspondingly harder to prove.

Theorem *If* F *and* G *are newforms for* $\Delta_0(a)$ *and* $\Delta_0(b)$ *respectively, then either* $F = G$ *and* $a = b$, *or else* F *and* G *have different eigenvalues for infinitely many* T_π *with* $\pi \nmid ab$.

§3.4 Cusp forms, Homology, and Modular Symbols

If Γ is a discrete subgroup of $GL(2,K)$ then denote by X_Γ the quotient topological space $\Gamma \setminus H_3$, and by $\overline{X}_\Gamma$ its closure $\Gamma\backslash H_3^*$. In general X_Γ is not a manifold, because Γ may contain elements of finite order; but if Γ contains no such points, then X_Γ is a real analytic Riemannian manifold (c.f. Kurčanov [8]).

If Γ does have elements of finite order, let $\Gamma' < \Gamma$ be a normal subgroup of finite index in Γ with no such elements. Then $X_{\Gamma'}$ is a manifold, and the finite quotient group $\overline{\Gamma} := \Gamma/\Gamma'$ acts on it. We have a map

$$\pi \ : \ X_{\Gamma'} \ \to \ X_\Gamma$$

induced by the identity on H_3, and this induces maps

$$\pi_* \ : \ H_1(\overline{X}_{\Gamma'}\ , Q) \ \to \ H_1(\overline{X}_\Gamma\ , Q)$$

and $$\pi^* \ : \ H^1(\overline{X}_\Gamma\ , Q) \ \to \ H^1(\overline{X}_{\Gamma'}, Q).$$

Lemma 3.4.1 *π_* is surjective and π^* is injective.*

Proof: (c.f. [8] Lemma 1). In the case of homology, the set $P_1 = \overline{X}_\Gamma \setminus X_\Gamma$ is finite, and the set $P_2 = \{\pi(P) : P \in X_{\Gamma'} \text{ is fixed by } \overline{\Gamma}\}$ is a one-dimensional submanifold of $\overline{X}_\Gamma$ (c.f. §2.2). It is clear that, given any path in $\overline{X}_\Gamma$ we can find a homologous path lying entirely within $\overline{X}_\Gamma \setminus (P_1 \cup P_2)$. If the path is closed, some multiple of it will lift to a closed path in $\overline{X}_{\Gamma'}$ since $\overline{\Gamma}$ is finite.

We have $H^1(\overline{X}_{\Gamma'}\ , C) \simeq \Omega/\Omega_0$, where Ω is the space of closed C^∞ 1-forms of compact support on $X_{\Gamma'}$, and Ω_0 is the subspace of forms of the form dF where F is a function on $X_{\Gamma'}$. If $\Omega^{\overline{\Gamma}}$ and $\Omega_0^{\overline{\Gamma}}$ denote the subspaces of Ω and Ω_0 consisting of forms which are invariant under the action of $\overline{\Gamma}$, then in (3.4.1) the image of π^* is clearly $\Omega^{\overline{\Gamma}} / \Omega_0^{\overline{\Gamma}}$.

In the case $\Gamma = \Delta_0(a)$, let $X_0(a) = X_{\Delta_0(a)}$. Kurčanov, in [8], proves that the map

$$S(a) \ \to \ H^1(\overline{X_0(a)}\ , C)\ ,$$

given by $$F \ \to \ F.\beta \quad ,$$

is an isomorphism. In fact, Kurčanov, and others, work with the larger group $\Delta_0^+(a) := \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL(2,K) : c \in a\}$. In the case where K has class number one the corresponding projective groups satisfy $[\overline{\Delta_0^+(a)} : \overline{\Delta_0(a)}] = 2$ since

$$\overline{\Delta_0^+(a)} = \overline{\Delta_0(a)} \cup \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}\overline{\Delta_0(a)} .$$

Hence on the right-hand side of the above isomorphism, one is restricting to forms invariant under $(z,t) \to (-z,t)$; similarly, on the left-hand side, one is restricting to the subspace $S^+(a)$ defined in the previous section.

Now there is an exact duality

$$\begin{array}{lcl} H^1(\overline{X}_{\Gamma'}, C) \times H_1(\overline{X}_{\Gamma'}, C) & \to & C \\ \text{given by} \quad (\omega , \gamma) & \to & \int_\gamma \omega . \end{array}$$

In this pairing the action of GL(2) on differential forms corresponds to the action on homology induced by the action on the points of H_3, since if $\alpha \in GL(2,C)$ and γ is a path representing a homology cycle, then

$$\int_{\gamma(\alpha)} F.\beta = \int_\gamma (F\circ\alpha).(\beta\circ\alpha) = \int_\gamma (F|\alpha).\beta$$

by definition of $F|\alpha$.

In this duality, restricting on the left to forms invariant under $\overline{\Gamma}$, that is to $\Omega^{\overline{\Gamma}}/\Omega_0^{\overline{\Gamma}}$, corresponds on the right to factoring out by the kernel of π_*. So we also have an exact duality between $H^1(\overline{X}_\Gamma , C)$ and $H_1(\overline{X}_\Gamma , C)$. In the case $\Gamma = \Delta_0(a)$, we can use Kurčanov's isomorphism to yield an isomorphism

$$S(a) \xrightarrow{\simeq} H_1(\overline{X_0(a)} , C).$$

In the next chapter I will show how to compute $V(a) := H_1(\overline{X_0(a)},Q)$ explicitly for any ideal a of O_K when K is one of the five Euclidean fields. We will be able to compute explicitly the action of:

the main involution J;

the W_π involutions for $\pi \mid a$;

the T_π for any $\pi \nmid a$.

It follows from the results of the previous section that we can find a basis for V(a) with respect to which each of the above operators acts with a diagonal matrix: this basis will be explicitly represented in terms of cycles on $\overline{X_0(a)}$. Old forms can be recognized, while the coefficients of newforms can be computed from their Hecke eigenvalues, as indicated in the previous section.

Modular Symbols Modular symbols are a convenient form of notation with which to calculate the homology of spaces $\overline{X_\Gamma}$ for various subgroups Γ. It seems appropriate to define them and give their basic properties here, although we will not use them until the following Chapter. They are discussed in the rational case by, for example, Manin in [10] or Lang in his book [9]. Kurčanov has also used them for complex quadratic fields, in work relating to the Birch - Swinnerton Dyer conjecture, for which they were originally invented by Birch.

Let A and B be two points in the extended upper half-space $H_3^* = H_3 \cup K \cup \{j\infty\}$ which are equivalent under the action of Γ: so there exists $\gamma \in \Gamma$ such that $\gamma(A) = B$. Then any smooth path from A to B in H_3 projects to a closed path in the quotient space $\overline{X}_\Gamma = \Gamma \setminus H_3^*$, whose homology class in $H_1(\overline{X}_\Gamma , Z)$ depends only on A and B and not on the path chosen (because H_3^* is simply connected). Denote this homology class by $\{A,B\}_\Gamma$, or simply by $\{A,B\}$ if the group Γ is clear from the context. If we identify homology classes with functionals on the space of differentials, then we may extend this definition to points A and B not equivalent under Γ: denote by $\{A,B\}$ the real homology class identified with the functional $\omega \to \int_A^B \varphi^* \omega$, where ω is a differential on $\overline{X}_\Gamma$ and $\varphi: H_3^* \to \overline{X}_\Gamma$ is the natural projection.

Modular symbols $\{A,B\}$ have the following properties, whose proof is immediate.

(3.4.2) (i) $\{A,A\} = 0$;

(ii) $\{A,B\} + \{B,A\} = 0$;

(iii) $\{A,B\} + \{B,C\} + \{C,A\} = 0$;

(iv) $\{\gamma A,\gamma B\}_\Gamma = \{A,B\}$ if $\gamma \in \Gamma$;

(v) $\{A,\gamma A\}_\Gamma = \{B,\gamma B\}_\Gamma$ if $\gamma \in \Gamma$, for any A and B in H_3^*;

(Proof:
$$\begin{aligned}\{A,\gamma A\} &= \{A,B\} + \{B,\gamma B\} + \{\gamma B,\gamma A\} && \text{by (ii) and (iii)}\\ &= \{A,B\} + \{B,\gamma B\} + \{B,A\} && \text{by (iv)}\\ &= \{B,\gamma B\} && \text{by (ii)}\end{aligned}$$
).

(vi) $\{A,\gamma A\}_\Gamma \in H_1(\overline{X}_\Gamma , Z)$ if $\gamma \in \Gamma$.

An elementary geometrical argument similar to the one used in the proof of Lemma 3.4.1 shows that any element of $H_1(\overline{X}_\Gamma , Z)$ can in fact be written as $\{A,\gamma A\}$ for some $\gamma \in \Gamma$, and $A \in P_1 = K \cup \{j\infty\}$.

In the next Chapter we will see that in fact the rational homology can be generated by elements of the form $\{\gamma(0),\gamma(j\infty)\}_\Gamma$ for $\gamma \in SL(2,O_K)$.

§3.5 Periods of Cusp Forms

Recall that in the rational case, if $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ is a newform for $\Gamma_0(N)$ (so that $a_1 = 1$ and $T_p f = a_p f$ for all $p \nmid N$) with all its Fourier coefficients a_n rational, then in the homology of $\Gamma_0(N) \setminus H^*$ there are two one-dimensional eigenspaces with eigenvalues a_n, one with eigenvalue +1 and one with eigenvalue −1 for the conjugation involution $z \to -\bar{z}$; and that the periods of the associated form $2\pi i f(z)$, integrated around the corresponding cycles, are the periods of an elliptic curve uniformized by functions on $X_0(N)$, with zeta-function $\Sigma a_n n^{-s}$, and this curve (conjecturally) has conductor N. For $N \leq 330$ the appropriate calculations have been carried out by Tingley, as described in his thesis [23].

One might hope that in the complex quadratic case the periods of the differential $F.\beta$, where F is a cusp form, might also be of interest (c.f. Weil [24] and [25], last paragraph). Unfortunately, the symmetry induced by complex conjugation in the rational case, which led to a pair of periods being associated to each newform, is not now available; although there is some symmetry between the spaces $S^+(a)$ and $S^-(a)$, defined in §3.3, as will be demonstrated in a later Chapter. However, the Fourier expansions introduced in §3.2 do enable us to compute a single period of a differential $F.\beta$, given sufficiently many of its coefficients, fairly rapidly and accurately; so we record the formulae here, and in Chapter 5 will give the approximate periods for some of the newforms given there in the tables.

First recall from the first section of this Chapter that Hankel's function K_0 satisfies the differential equation

$$tK_0''(t) + K_0'(t) - tK_0(t) = 0,$$

which can be written

$$-tK_0(t) = -\frac{d}{dt}(tK_0'(t)) = \frac{d}{dt}(tK_1(t)) \tag{3.5.1}$$

since $K_1(t) = -K_0'(t)$

Suppose that $F(z,t)$ is given by a Fourier expansion (3.2.13) and we wish to know its integral over some closed cycle γ in $\overline{X_0(a)}$; by the previous section, we can write this cycle in the form $\{\alpha, j\infty\}$ where $\alpha \in K$ is equivalent to $j\infty$ under $\Delta_0(a)$. Choosing a vertical path from $(\alpha,0)$ to $j\infty$ in H_3^*, the dz and $d\bar{z}$ components of $\beta = (-\frac{dz}{t}, \frac{dt}{t}, \frac{d\bar{z}}{t})$ vanish, giving $\int_\gamma F.\beta = \int_0^\infty F_1(\alpha,t)\frac{dt}{t}$ where $F = (F_0,F_1,F_2)$. Substituting the Fourier expansion (3.2.13) for F gives

$$(3.5.2) \qquad \int_\gamma F.\beta = \int_0^\infty \sum_{\xi\in 0_K} c(\xi) tK_0(4\pi|\eta^{-1}\xi|\, t)\psi(\eta^{-1}\xi\alpha)dt$$
$$= \sum_{\xi\in 0_K}\left(c(\xi)\psi(\eta^{-1}\xi\alpha)\int_0^\infty tK_0(4\pi|\eta^{-1}\xi|t)dt\right)$$
$$= K\sum_{\xi\in 0_K}\left(\frac{c(\xi)}{|\xi|^2}\psi(\eta^{-1}\xi\alpha)\right), \text{ where}$$
$$K = \frac{|\eta|^2}{(4\pi)^2}\int_0^\infty tK_0(t)dt$$

is a constant. But although convergence of the above sum is assured by the estimate $c(\xi) = O(|\xi|^{-a})$ for some $a \geq 0$, in practice it converges very slowly. This can be remedied as follows.

Let $g \in \Delta_0(a)$ be such that $g(j\infty) = \alpha$. Then the cycle $\{\alpha, j\infty\} = \{g(j\infty), j\infty\}$ can be represented by $\{g(P),P\}$ for any $P \in H_3$, by property (v) of (3.4.2). Since $\{g(P),P\} = \{g(P),j\infty\} - \{P,j\infty\}$ it is enough to consider cycles γ of the form $\{P,j\infty\}$. If $P = (\alpha,t_0)$ where $t_0 > 0$, then (3.5.2) becomes

$$\int_\gamma F.\beta = \frac{|\eta|t_0}{(4\pi)}\sum_{\xi\in 0_K}\left(c(\xi)|\xi|^{-2}\psi(\eta^{-1}\alpha\xi)\int_{4\pi|\xi\eta^{-1}|t_0}^\infty tK_0(t)dt\right).$$

But from (3.5.1), since $K_0(t)$ and $K_1(t)$ decrease rapidly to 0 as $t \to \infty$, we have

$$\int_{t_0}^\infty tK_0(t)dt = -t_0K_1(t_0). \qquad \text{Hence}$$

$$(3.5.3) \qquad \int_\gamma F.\beta = (-)\frac{|\eta|t_0}{4\pi}\sum_{\xi\in 0_K}\left(c(\xi)|\xi|^{-1}\psi(\eta^{-1}\alpha\xi)K_1(4\pi|\xi\eta^{-1}|t_0)\right).$$

Since $K_1(t)$ becomes very small very quickly (faster than e^{-t}) this sum converges very fast: and the larger the value of t_0, the faster it will converge. If $g = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix}$ then it is best to take $P = (-\frac{d}{Nc}, \frac{1}{|Nc|})$ where $(N) = a$, for then $Q = g(P) = (\frac{a}{Nc}, \frac{1}{|Nc|})$, so that in calculating both the integrals, over $\{Q, j\infty\}$ and $\{P, j\infty\}$, we can take $t = |Nc|^{-1}$. In practice one has considerable choice in which matrix g to use: so one chooses one with $|c|$ as small as possible.

CHAPTER 4

Modular Symbols and the Calculation of Homology

In this Chapter, I will show how to calculate $H_1(G\backslash H_3^*, Q)$ where G is a subgroup of finite index in $\Delta = SL(2,O_K)$ and K is one of the five Euclidean fields $Q(\sqrt{-d})$ for d = 1,2,3,7, 11. Some remarks will be made at the end about how to generalize the procedure to the other fields with class number one (d = 19,43,67 and 163) and the fields with class number greater than one.

The main features of the algorithm are identical for the five fields, although there are of course differences in detail. The algorithm is an extension of the one given by Manin in [10] for subgroups of the modular group SL(2,Z) which I will describe briefly in §4.1: I will borrow some of Manin's notation, but present a somewhat simpler version of the proof. Then, in §4.2, I talk about the algorithm for the five Euclidean quadratic fields, giving the plan of approach which will be common to the five fields. The five fields are then dealt with in turn. The geometry developed in Chapter 2, and in particular the fundamental regions described there, will play an important part in this discussion.

§4.1 Review of the Algorithm for subgroups of SL(2,Z)

Let G be a subgroup of finite index in $\Gamma = SL(2,Z)$; we wish to calculate $H_1(G\backslash H^*, Q)$. As a fundamental region for Γ on H we will use the triangular region F with vertices at $i\infty$, 0, and $\rho = \frac{1}{2}(1+\sqrt{-3})$ (see Diagram 4.1). The $\{0,i\infty\}$ edge of F is self-identified by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. (The orientation is reversed: S interchanges 0 and $i\infty$ while fixing i). The other edges are identified by TS (recall $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so that $TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$) which fixes ρ and maps $0 \to i\infty \to 1 \to 0$. So

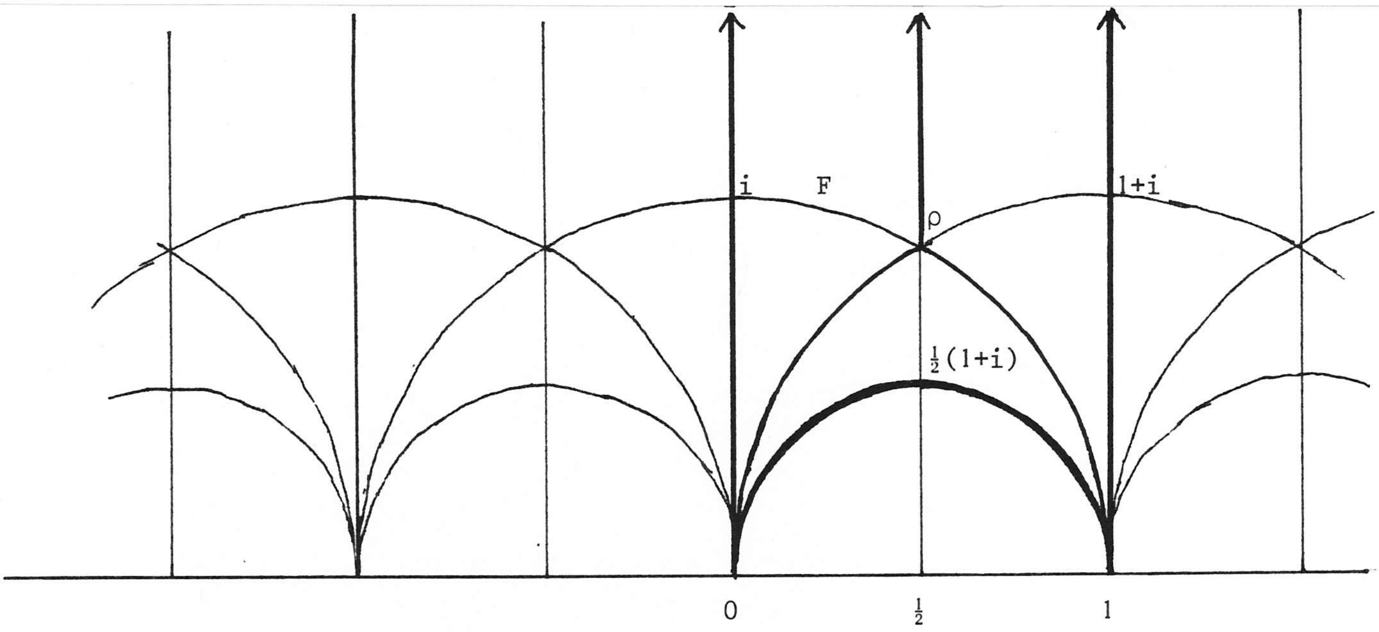


Diagram 4.1

the transforms of F under $\{I, TS, (TS)^2\}$ cover a larger triangle F+ with vertices at $i\infty$, 0, and 1. (See Diagram 4.1).

Write $\Gamma = \bigcup_{j=1}^{k} G\gamma_j$ where $\{\gamma_j\}_{j=1}^{k}$ is a set of right coset representatives for G in Γ. Then a fundamental region for G in H^* is given by $\bigcup_{j=1}^{k} \gamma_j F$ since $G(\cup\gamma_j F) = \cup(G\gamma_j)F = \Gamma F = H^*$. Also the tesselation of the upper half-plane with the transforms of F clearly gives a triangulation of $G\backslash H^*$ under the natural projection $H^* \to G\backslash H^*$. So one could calculate the homology of $G\backslash H^*$ by taking as generators the three edges of $\gamma_j F$ for each $j = 1,2,\ldots,k$, and relations of two types: one type recording that the sum of the three edges of each $\gamma_j F$ is zero, and the other recording the 'gluing together' of adjacent $\gamma_j F$. However it is simpler if we unite the $\gamma_j F$ in sets of three as in the diagram: let $F+ = F \cup (TS)F \cup (TS)^2 F$ so that F+ has edges $\{0, i\infty\}$, $\{i\infty, 1\} = \{TS(0), TS(i\infty)\}$, and $\{1, 0\} = \{(TS)^2(0), (TS)^2(i\infty)\}$. Then we can forget about the inner edges entirely, and take as generators the single edge $\{\gamma_j(0), \gamma_j(i\infty)\}$ for each coset representative γ_j; relations of the first kind

$$\{\gamma_j(0), \gamma_j(i\infty)\} + \{\gamma_j(i\infty), \gamma_j(1)\} + \{\gamma_j(1), \gamma_j(0)\} = 0$$

for each coset representative γ_j; and relations of the second kind

$$\{\gamma_j(0),\gamma_j(i\infty)\} + \{\gamma_j(i\infty),\gamma_j(0)\} = 0.$$

Writing (γ) for the edge $\{\gamma(0),\gamma(i\infty)\}_G$ these can be written concisely as

(4.1.1) (i) $$(\gamma) + (\gamma TS) + (\gamma(TS)^2) = 0;$$

(ii) $$(\gamma) + (\gamma S) = 0.$$

If $G\gamma = G\gamma TS$ then (i) is to be interpreted as $(\gamma) = 0$, and similarly if $G\gamma = G\gamma S$ then (ii) is to be interpreted as $(\gamma) = 0$: this is because we are calculating the rational homology, which is torsion-free.

Writing $[\alpha]$ for the equivalence class of cusps modulo G containing $\alpha \in \mathbf{Q} \cup \{i\infty\}$, we can write the boundary map ∂ from the space of 1-cycles to the space of 0-cycles as

(4.1.2) $$\partial(\gamma) = [\gamma(i\infty)] - [\gamma(0)],$$

extended by linearity.

Hence we have the following result.

<u>Theorem 4.1.3</u> *Form the* $\mathbf{Q}$*-vector space with symbols* (γ) *as basis, for a complete set of coset representatives* γ *of* G *in* Γ, *modulo all relations of the form* (4.1.1)(i),(ii). *Let* H(G) *denote the kernel of the homomorphism* ∂ *(defined by* (4.1.2) *and extended by linearity). Then the map*

(4.1.4) $$(\gamma) \to \{\gamma(0),\gamma(i\infty)\}_G$$

gives an isomorphism from H(G) *to* $H_1(G\backslash H^*, \mathbf{Q})$.

<u>Note</u>: The boundary map ∂ is well-defined since each of the relations (4.1.1) has 'boundary 0'; that is, $\partial(\gamma) + \partial(\gamma S) = 0$ and $\partial(\gamma) + \partial(\gamma TS) + \partial(\gamma(TS)^2) = 0$.

Notice that we are generating the homology entirely by paths whose end-points are cusps: more strongly, we are only using paths of the form $\{\gamma(0),\gamma(i\infty)\}$ for $\gamma \in \Gamma$. In fact there is a simple way of expressing any path between cusps as a sum of such paths. It suffices to do so for paths of the form $\{0,\frac{a}{b}\}$ for $\frac{a}{b} \in \mathbf{Q}$, a fraction in its lowest terms. Write down the continued fraction convergents of $\frac{a}{b}$:

$$\frac{a}{b} = \frac{a_n}{b_n},\ \frac{a_{n-1}}{b_{n-1}},\ \ldots,\ \frac{a_0}{b_0} = \frac{a_0}{1},\ \frac{a_{-1}}{b_{-1}} = \frac{1}{0},\ \frac{a_{-2}}{b_{-2}} = \frac{0}{1},$$

(the last terms are added formally for convenience). Then, as is well known, $a_k b_{k-1} - a_{k-1} b_k = (-1)^{k-1}$, so that

$$\gamma_k := \begin{pmatrix} a_k & (-1)^{k-1} a_{k-1} \\ b_k & (-1)^{k-1} b_{k-1} \end{pmatrix} \in \Gamma .$$

Then

$$(4.1.5) \qquad \{0,\tfrac{a}{b}\} = \sum_{k=-1}^{n} \left\{ \frac{a_{k-1}}{b_{k-1}}, \frac{a_k}{b_k} \right\} = \sum_{k=-1}^{n} \{\gamma_k(0), \gamma_k(i\infty)\} ,$$

as required.

This device is important in the actual computations, since we have a definition of various operators (Hecke operators T_p, and involutions of various kinds) defined on points, and hence on homology and modular symbols $\{A,B\}_G$, but not directly on the symbols (γ); so we need to pass from one to the other, and this is achieved by (4.1.4) and (4.1.5).

Two other ingredients are needed in order to make Theorem 4.1.3 into an algorithm: we need to give a set of coset representatives for G in Γ (and to determine to which coset any particular element of Γ belongs); and we need to decide when two cusps are equivalent under G. This will now be done explicitly for $G = \Gamma_0(N)$.

Manin [10] gives a set of symbols, which I will call M-symbols, which are in one-one correspondance with the cosets of $\Gamma_0(N)$ in Γ, as follows. Consider the set of all ordered pairs (c,d) where $c,d \in Z$ and h.c.f.$(c,d,N) = 1$; call two such pairs (c_1,d_1) and (c_2,d_2) <u>equivalent</u> if there exists $u \in Z$ with h.c.f.$(u,N) = 1$ such that $c_1 \equiv uc_2$ and $d_1 \equiv ud_2 \pmod N$; denote the equivalence class of (c,d) by $(c:d)$ and the set of such classes by $P^1(N)$ (the 'projective line' over the ring $Z/(N)$). We can map $\Gamma \to P^1(N)$ as follows:

$$(4.1.6) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (c:d).$$

A simple computation shows that this map is constant on right cosets of

$\Gamma_0(N)$ in Γ. It is surjective, since given c,d in $\mathbb{Z}$ with $(c,d,N) = 1$ there exist a,b in $\mathbb{Z}$ such that $ad-bc \equiv 1 \pmod N$; by (2.5.1) we can find a',b',c' and d' congruent to a,b,c, and d (mod N) respectively, such that $a'd' - b'c' = 1$; and then $(c':d') = (c:d)$ is the image of $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. A similar argument shows that the map is injective.

Moreover, the map (4.1.6) preserves the right coset action of Γ on the coset space $[\Gamma:G]$ provided that we define

(4.1.7)
$$(c:d)\begin{pmatrix} p & q \\ r & s \end{pmatrix} = (cp + dr : cq + ds),$$

since
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

for $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma$. This makes M-symbols very easy and convenient to work with: as in the above proof we can always assume of a symbol (c:d) that $(c,d) = 1$; then if a and b are any integers such that $ad-bc = 1$, we may identify (c:d) with $\{\frac{b}{d}, \frac{a}{c}\}_G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}(0), \begin{pmatrix} a & b \\ c & d \end{pmatrix}(i\infty)\}$, and so $\partial((c:d)) = \left[\frac{a}{c}\right] - \left[\frac{b}{d}\right]$. Note that a is only determined modulo c and b modulo d, so that the fractions $\frac{a}{c}$ and $\frac{b}{d}$ are only determined modulo integers: this is consistent, since $\Gamma_0(N)$ always contains $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so that cusps whose difference is an integer are equivalent modulo $\Gamma_0(N)$ for every N.

In terms of M-symbols, conversion formula (4.1.5) reads

(4.1.8)
$$\{0, \frac{a}{b}\} = \sum_{k=-1}^{n} (b_k : (-1)^{k-1} b_{k-1})$$

Note the alternating sign, and also that only the denominators of the successive convergents appear in the formula.

As for equivalence of cusps, we have the following result.

<u>Proposition 4.1.9</u> *Let* $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$ *be written in their lowest terms. Then the following are equivalent:*

(i) *There exists* $\gamma \in \Gamma_0(N)$ *such that* $\gamma(\frac{p_1}{q_1}) = \frac{p_2}{q_2}$;

(ii) $s_1 q_2 \equiv s_2 q_1$ (modulo $(q_1 q_2, N)$) *where* $p_j s_j \equiv 1$ (modulo q_j) *for* $j = 1,2$.

Proof: For $j = 1,2$ choose r_j and s_j such that $p_j s_j - r_j q_j = 1$; then $\gamma_1 = \begin{pmatrix} q_1 & -p_1 \\ s_1 & -r_1 \end{pmatrix} \in \Gamma$ and $\gamma_1(\frac{p_1}{q_1}) = 0$, while $\gamma_2 = \begin{pmatrix} -r_2 & p_2 \\ -s_2 & q_2 \end{pmatrix} \in \Gamma$ and $\gamma_2(0) =$

$\frac{p_2}{q_2}$. Any element in Γ fixing 0 has the form $\begin{pmatrix} \pm 1 & 0 \\ x & \pm 1 \end{pmatrix}$ for some $x \in Z$, and we can ignore the possibility of minus signs since $\begin{pmatrix} -1 & 0 \\ x & -1 \end{pmatrix}$ gives the same transformation as $\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}$. So the general element in Γ sending $\frac{p_1}{q_1}$ to $\frac{p_2}{q_2}$ is

$$\gamma_2 \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \gamma_1 = \begin{pmatrix} * & * \\ xq_1q_2 + s_1q_2 - q_1s_2 & * \end{pmatrix},$$

and thus (i) holds if and only if we can solve the congruence

$$(4.1.10) \qquad xq_1q_2 + s_1q_2 - q_1s_2 \equiv 0 \pmod N$$

for x, which is if and only if (ii) holds.

Note that (4.1.10) implies that $(N,q_1) = (N,q_2)$ so this condition, decidable at a glance, is necessary for $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ to be equivalent.

§4.2 The Algorithm for Complex Quadratic Fields: Common Features

In this section Δ will denote $SL(2,O_K)$, where K is any one of the fields $Q(\sqrt{-d})$ for d = 1, 2, 3, 7, or 11, and G will denote a subgroup of finite index in Δ.

Our plan for each field is as follows: choose a fundamental region F for the action of Δ on H_3^*, with $\{0,j\infty\}$ as one of its edges; form a larger basic polyhedron by taking the union of the transforms of F by a finite subgroup G_P which stabilizes a vertex P of F (compare Diagram 4.1 where the vertex ρ is stabilized by the subgroup of order 3, generated by TS). In the simplest cases the only edges of the basic polyhedron $\overline{F}$ will be the transforms of $\{0,j\infty\}$, one for each element of the finite group G_P. We will have 'face relations' to replace (4.1.1)(i), usually of two types; and 'edge relations' to replace (4.1.1)(ii), to record the juxtaposition of the transforms of the basic polyhedron: these latter relations can clearly be found by considering the transforms which share the edge $\{0,j\infty\}$. Then a Theorem similar to 4.1.3 will hold, for G a subgroup of finite index in Δ, with those new relations to replace (4.1.1)(i),(ii), between symbols (γ) corresponding to the edge $\{\gamma(0),\gamma(j\infty)\}$

for each coset representative γ of G in Δ. The boundary map ∂ is given by

(4.2.1) $$\partial(\gamma) = [\gamma(j\infty)] - [\gamma(0)]$$

where $[\alpha]$ now denotes the equivalence class of the cusp $\alpha \in K \cup \{j\infty\}$ under the action of G.

As before we can express any path $\{\alpha,\beta\}$ between cusps α and β (in K) as a sum of paths of the form $\{\gamma(0),\gamma(j\infty)\}$, using the continued fraction expansions of α and β. We can do this because K is Euclidean, so that for any $\alpha \in K$ there exists $\xi \in O_K$ such that $|\alpha - \xi| < 1$, where $|\ |$ denotes ordinary complex absolute value; hence we can define finite continued fraction expansions for elements of K just as for rationals. The successive convergents have the same properties as before, and the analogue of (4.1.5) clearly holds.

The definition of M-symbols as coset representatives for the particular subgroups $\Delta_0(a)$, where a is an ideal of O_K, generalizes with no difficulty: we let $P^1(a)$ be the set of symbols $(c:d)$ where $c,d \in O_K$ and $(c,d) + a = O_K$, with $(c_1:d_1) = (c_2:d_2)$ whenever there exists $u \in O_K$ such that
$(u) + a = O_K$;
$c_1 - uc_2$, $d_1 - ud_2 \in a$.
Then (4.1.6) gives a map from Δ to $P^1(a)$ which induces an isomorphism of right Δ-spaces from $[\Delta:\Delta_0(a)]$ to $P^1(a)$, where the action of Δ on $P^1(a)$ is again given by (4.1.7).

The condition for cusps to be equivalent under $\Delta_0(a)$ is slightly stronger than (4.1.9)(ii) when O_K contains 'extra' units, namely when $K = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$. For, the general element of Δ fixing 0 is now $\begin{pmatrix} u & 0 \\ x & u^{-1} \end{pmatrix}$ where $u \in O_K^*$. This last matrix has the same action as $\begin{pmatrix} u^2 & 0 \\ ux & 1 \end{pmatrix}$. Inspection of the proof of (4.1.9) now yields

<u>Proposition 4.2.2</u> *Let* $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in K$ *be written in their lowest terms. Then the following are equivalent:*

(i) *There exists* $\gamma \in \Delta_0(a)$ *such that* $\gamma(\frac{p_1}{q_1}) = \frac{p_2}{q_2}$;

(ii) *There exists* $u \in O_K^*$ *such that*

$$s_1q_2 \equiv u^2 s_2 q_1 \quad (\text{modulo } (q_1q_2) + a)$$

where $p_k s_k \equiv 1$ (modulo (q_k)) *for* $k = 1,2$.

Note: If $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ then $u^2 = 1$ for all $u \in O_K^*$ so that condition (ii) may be replaced by the simpler

(ii)' $s_1q_2 \equiv s_2q_1$ (modulo $(q_1q_2) + a$).

If $K = \mathbb{Q}(\sqrt{-1})$ then $u^2 = \pm 1$, and if $K = \mathbb{Q}(\sqrt{-3})$ then $u^2 = 1$, ω, or ω^2 where $\omega = \frac{1}{2}(-1 + \sqrt{-3})$.

The main involution J of 3.3 is given by

$$(4.2.3) \qquad (\gamma) \to \left(\begin{pmatrix}\varepsilon^{-1} & 0\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}\right)$$

which in terms of M-symbols becomes

$$(4.2.4) \qquad (c:d) \to (\varepsilon c:d)$$

where ε is a generator of the unit group O_K^*. To calculate the action of the Hecke operators and W-involutions we first convert to modular symbols via (4.1.4):

$$(4.2.5) \qquad (\gamma) \to \{\gamma(0), \gamma(j\infty)\}.$$

For the Hecke operators T_π, where π is a prime element, we then use the formula

$$(4.2.6) \qquad T_\pi : \{\alpha, \beta\} \to \sum_{\xi \bmod (\pi)} \left\{\frac{\alpha + \xi}{\pi}, \frac{\beta + \xi}{\pi}\right\} + \{\pi\alpha, \pi\beta\}$$

obtained from (3.3.5). Similarly from (3.3.10), for the involution W_π we use the formula

$$(4.2.7) \qquad W_\pi : \{\alpha, \beta\} \to \left\{\frac{\pi^r x\alpha + y}{Nz\alpha + \pi^r w}, \frac{\pi^r x\beta + y}{Nz\beta + \pi^r w}\right\}$$

where $a = (N)$, and r is the highest power of π dividing N, and x,y,z, and w are chosen so that

$$\pi^{2r}xw - Nzy = \pi^r.$$

Lastly we reconvert all the modular symbols appearing on the right of formulae (4.2.6) and (4.2.7) to linear combinations of M-symbols, via (4.1.8).

Let $V(a) = H_1(\Delta_0(a) \setminus H_3^*, \mathbb{Q})$. We can decompose $V(a)$ according to the eigenvalues of J as

$$V(a) = V^+(a) \oplus V^-(a)$$

where J acts as $+1$ on $V^+(a)$, and as -1 on $V^-(a)$. Now as abstract vector spaces we have

$$V^+(a) \simeq V(a) / V^-(a)$$

and $V^-(a) = \{x-xJ : x \in V(a)\}$. In practice it is often convenient to calculate $V^+(a)$ in this way, by including extra relations of the form

(4.2.8) $$(\gamma) = J(\gamma)$$

or, in terms of M-symbols,

(4.2.8)' $$(c:d) = (\varepsilon c:d).$$

The advantage is that by means of (4.2.8) or (4.2.8)' we can halve the number of symbols we have to work with, which means that the time taken for certain stages of the calculation is reduced by a factor of four; the amount of space is also reduced, which can be an important consideration when Na is large. Also, by means of (4.2.8) we can often simplify the other relations, thus saving time again. This will be seen in particular cases in the following sections of this Chapter.

Care must be taken in calculating ∂ under this scheme: since ∂ does not annihilate all of $V^-(a)$, it does not induce a well-defined map on the quotient $V(a)/V^-(a)$. However, a moment's thought shows that we can avoid this by calculating the kernel of $\partial + \partial J$ instead of ∂, since the map $1 + J$ projects $V(a)$ onto $V^+(a)$.

Similarly we can calculate $V^-(a)$ as $V(a)/V^+(a)$ if we include the extra relations

(4.2.9) $$(\gamma) = -J(\gamma), \quad \text{or}$$

(4.2.9)' $$(c:d) = -(\varepsilon c:d),$$

and restrict to the kernel of $\partial - \partial J$ instead of the kernel of ∂.

§4.3 The Algorithm for $\mathbb{Q}(\sqrt{-1})$

Let $i = \sqrt{-1}$ and $\Delta = SL(2, Z[i])$. Recall from §2.4 that $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. We replace half the fundamental region D given there by its image under S to get a fundamental region F for Δ which is spike-shaped (see Diagram 4.2) with vertices at 0, ∞, and at three points on the unit sphere: $P_1 = (\frac{1}{2}, 0, \frac{1}{2}\sqrt{3})$, $P_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2})$ and $P_3 = (0, \frac{1}{2}, \frac{1}{2}\sqrt{3})$. The lower curved faces are parts of spheres, and they meet along a common circular arc between 0 and P_2. The stabilizer of edge P_1P_2 is of order 3, generated by TS. The stabilizer of edge P_3P_2 is of order 3, generated by UR. The stabilizer of P_2 is the group G_P $= \langle TS, UR \rangle$ of order 12 (sketch proof below): so 12 transforms of F meet at P_2. The union of these 12 forms the 'basic polyhedron' F+ which is a (hyperbolic) octahedron, with four vertical faces meeting at ∞ (each subdivided by the edges of F and suitable transforms of F in the manner indicated in Diagram 4.1) and four curved faces meeting at $(\frac{1}{2}, \frac{1}{2}, 0)$. A sketch of F+, and a plan of its projection onto the 'floor' are given in Diagram 4.3. So there are twelve edges: four vertical half-lines meeting at ∞; four semicircles of diameter 1 in vertical planes, joining the vertices $0, 1, 1+i$, and i; and four semicircles of diameter $\frac{1}{2}\sqrt{2}$ meeting at $\frac{1}{2}(1+i)$. These edges are precisely the images of $\{0, \infty\}$ under the action of G_P.

Sketch proof: In order to determine the stabilizer of P_2 one proceeds as follows. The matrix $\begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}(1+i) \\ 0 & 1 \end{pmatrix}$ transforms j $(= (0,1))$ into P_2, while $\begin{pmatrix} 1 & -\frac{1}{2}(1+i) \\ 0 & \frac{1}{2}\sqrt{2} \end{pmatrix}$ does the reverse. The general element in $GL(2, \mathbb{C})$ fixing j has the form $\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$; so the general element in $GL(2, \mathbb{C})$ fixing P_2 is

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}(i+i) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2}(1+i) \\ 0 & \frac{1}{2}\sqrt{2} \end{pmatrix}$$

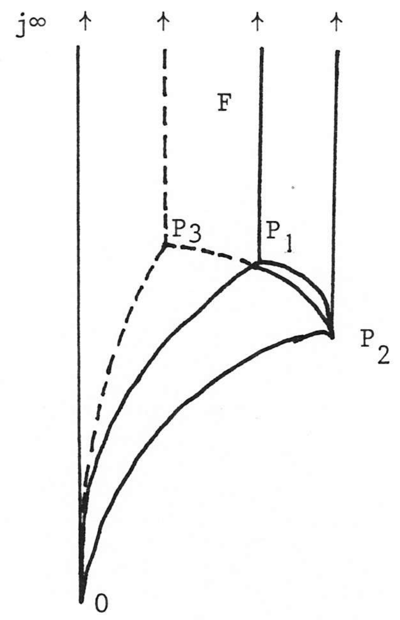
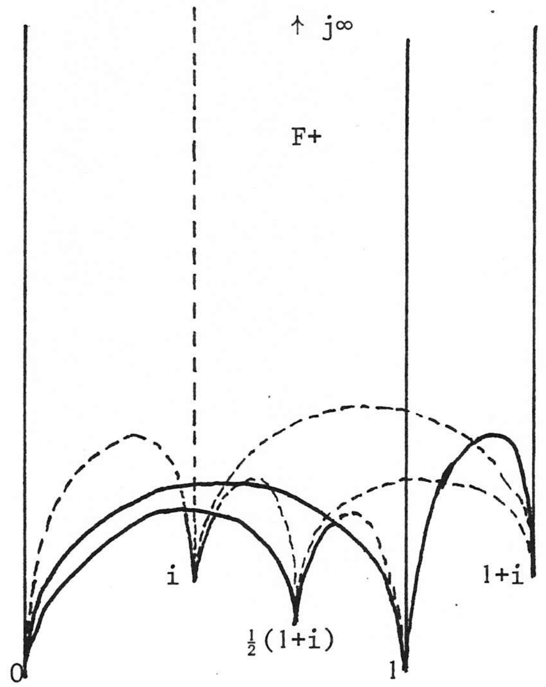
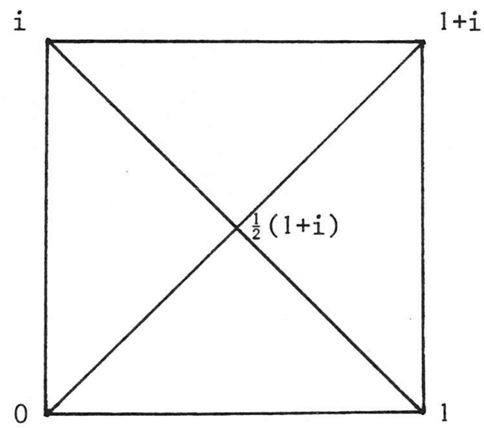


Diagram 4.2

Diagram 4.3



Plan:



$$= \begin{pmatrix} \frac{1}{2}u\sqrt{2} - \frac{1}{2}\bar{v}(1+i) & \frac{1}{4}\sqrt{2}(\bar{u} - u)(1+i) + \frac{1}{4}\bar{v}(1+i)^2 + \frac{1}{2}v \\ -\bar{v} & \frac{1}{2}\bar{v}(1+i) + \frac{1}{2}\,2\bar{u} \end{pmatrix} .$$

The conditions that this matrix should lie in $SL(2,O_K)$ are firstly that the entries should be in $Z[i]$, and secondly that the determinant should be 1. This second condition is $\frac{1}{2}(|u|^2 + |v|^2) = 1$. The first certainly implies that $v \in Z[i]$, and then also (by considering the upper left entry) that $w \in Z[i]$ where $w = (i-1)u/\sqrt{2}$. So we need to look at pairs of Gaussian integers (w,v) such that $|w|^2 + |v|^2 = 2$. Clearly we must have $(|w|^2,|v|^2) = (2,0)$, $(1,1)$, or $(0,2)$. This eventually leads to 24 pairs (w,v), and hence 24 elements of $SL(2,Z[i])$ which fix P_2, or 12 elements of $PSL(2,Z[i])$: they form the group $\langle TS,UR\rangle$ as stated above.

To determine the shape of the basic polyhedron, one first calculates the vertices of the 12 transforms of F under the action of G_P; then one glues together those transforms with a face in common; lastly, one ignores internal edges and vertices (as in Diagram 4.1 where three transforms of the triangle F are glued along common edges, and then the three internal edges and the vertex ρ are erased to give the larger triangle F+).

Similar computations have to be carried out for the other four fields: in subsequent sections, all such details are omitted for the sake of brevity.

So to generate the 1-homology of the quotient space of H_3^* by the action of a subgroup G of Δ, we take a symbol (γ) representing the path $\{\gamma(0),\gamma(\infty)\}$ for each matrix γ in a set of coset representatives for G in Δ. Relations between these are given first by the boundaries of the triangular faces of F+ and its transforms: these are of two types:

(4.3.1)(i) $\qquad (\gamma) + (\gamma TS) + (\gamma(TS)^2) = 0;$

(ii) $\qquad (\gamma) + (\gamma UR) + (\gamma(UR)^2) = 0.$

Secondly, observe that four copies of F+ meet at the edge $\{0,\infty\}$: its images under I and RS (with orientation preserved), and under R and S (with orientation reversed). This gives relations

(iii) $\qquad (\gamma) + (\gamma R) = 0;$

(iv) $\qquad (\gamma) + (\gamma S) = 0.$

The other relation, $(\gamma) = (\gamma RS)$, is a consequence of (iii) and (iv).

The boundary of (γ) is given by (4.2.1). So we have the following result.

<u>Theorem 4.3.2</u> *Form the* $\mathbb{Q}$*-vector space with symbols* (γ) *as basis, for a complete set of coset representatives* γ *of* G *in* Δ*, modulo all relations of the form (4.3.2)(i),(ii),(iii), and (iv). Let* H(G) *denote the kernel of the homomorphism* ∂ *defined by (4.2.1) and extended by linearity. Then the map*

$$(\gamma) \to \{\gamma(0),\gamma(j\infty)\}_G$$

gives an isomorphism from H(G) *to* $H_1(G\backslash H_3^*, \mathbb{Q})$.

Note: The boundary map ∂ is well-defined because each of the relations (4.3.1)(i) - (iv) has boundary zero: this is clear since TS permutes 0, ∞, and 1 cyclically; UR permutes 0, ∞, and i cyclically; while both R and S interchange 0 and ∞.

In the case $G = \Delta_0(\mathfrak{a})$ we can use M-symbols as coset representatives. The relations become

$$
\begin{aligned}
(4.3.3)(\text{i}) \qquad & (c:d) + (c+d:-c) + (d:-c-d) = 0;\\
(\text{ii}) \qquad & (c:d) + (ic+d:c) + (d:ic+d) = 0;\\
(\text{iii}) \qquad & (c:d) + (d:c) = 0;\\
(\text{iv}) \qquad & (c:d) + (-d:c) = 0.
\end{aligned}
$$

Notice that (iii) and (iv) together imply that $(c:d) = (-c:d)$ for all symbols $(c:d)$.

Let $V(\mathfrak{a}) = H_1(\Delta_0(\mathfrak{a})\backslash H_3^*, \mathbb{Q})$. Then Theorem 4.3.2 becomes

<u>Theorem 4.3.4</u> *Form the* $\mathbb{Q}$*-vector space with symbols* $(c:d)$ *as basis elements, for each* $(c:d) \in P^1(\mathfrak{a})$*, modulo all relations of the form (4.3.3)(i),(ii),(iii), and (iv). Let* $H(\mathfrak{a})$ *denote the kernel of the*

boundary map ∂, *where*

$$\text{(4.3.5)} \qquad \partial(c:d) = \begin{bmatrix} a \\ c \end{bmatrix} - \begin{bmatrix} b \\ d \end{bmatrix},$$

extended by linearity. Then

$$\text{(4.3.6)} \qquad (c:d) \to \{\tfrac{b}{d}, \tfrac{a}{c}\}$$

gives an isomorphism $H(a) \to V(a)$.

Note: In (4.3.5) and (4.3.6), a and b are any numbers in O_K chosen so that ad-bc = 1 (recall that in any symbol (c:d) we may assume that c and d are relatively prime).

Let $I_i = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$. Then we have the relations

$$I_i^{-1} S I_i = R$$

and

$$I_i T S I_i^{-1} = UR,$$

so that J sends relations of the form (4.3.3)(iii) into relations of the form (4.3.3)(iv), and relations of the form (4.3.3)(i) into relations of the form (4.3.3)(ii). Hence if we follow the 'short cut' scheme of §4.2 and introduce either the relation

$$(c:d) = (ic:d)$$

(for $V^+(a)$), or the relation

$$(c:d) = -(ic:d)$$

(for $V^-(a)$), then we may omit relations (ii) and (iv) altogether, leaving us with just three types of relation:

$$\text{(4.3.7)(i)} \qquad (c:d) = \pm(ic:d);$$

$$\text{(ii)} \qquad (c:d) + (-d:c) = 0;$$

$$\text{(iii)} \qquad (c:d) + (c+d:-c) + (d:-c-d) = 0.$$

Notice that the latter two are simply the 'rational relations' used in §4.1, coming from the relations $S^2 = (TS)^3 = I$ of the modular group $SL(2,Z)$. So the relations we have to apply to the symbols consist of the 'ordinary' modular relations, from $S^2 = (TS)^3 = I$, with some additional relations. Suppose that the ideal a is generated by an element $a+bi \in Z[i]$ with h.c.f.(a,b) = 1. Then $Z[i]/a \simeq Z/(a^2+b^2)$, and so $P^1(a) \simeq P^1(a^2+b^2)$. Hence in this case $H_1^+(\Delta_0(a)\backslash H_3^*, Q)$ is a quotient of $H_1^+(\Gamma_0(a^2+b^2)\backslash H^*, Q)$:

here the first superscript '+' denotes the homology invariant under the main involution $(z,t) \to (iz,t)$, while the second denotes the homology invariant under the conjugation involution $z \to -\bar{z}$. This is because the latter involution has the effect $(c:d) \to (-c:d)$ on symbols, and our relations (4.3.7)(i) imply that this action is trivial. So we have the following result.

Theorem *Let* $N \in \mathbb{Z}$ *be expressable as a sum of two squares,* $N = a^2 + b^2$, *with* $(a,b) = 1$. *Let* $\mathfrak{a}$ *be the ideal of* $\mathbb{Z}[i]$ *generated by* $a+bi$. *Then*

$$\dim S^{+}(\mathfrak{a}) \leq \dim S_2^{+}(\Gamma_0(N)).$$

Notes: (i) The condition on N is that it should not be divisible by 4 or by any prime $p \equiv 3 \pmod 4$.

(ii) The inequality is in fact very weak: examination of the results of computations of $\dim S^{+}(\mathfrak{a})$ given in Table 5.1.1 shows that the smallest N for which $\dim S^{+}((a+bi)) > 0$ is $N = 65$, whereas $\dim S_2^{+}(65) = 5$.

(iii) Similar remarks will hold for the other fields considered in the thesis: in each case, the relations between symbols will be obtained by adding one or more types of relation to the 'rational' relations.

We can sum up the results of this section in the following.

Theorem 4.3.8 *Form the* $\mathbb{Q}$*-vector space with symbols* $(c:d)$ *as basis elements, for each* $(c:d) \in P^1(\mathfrak{a})$*, modulo all relations of the form (4.3.7)(i), (ii) and (iii). Let* $H^{\pm}(\mathfrak{a})$ *denote the kernel of the map* $\partial^{\pm}$*, where*

$$\partial^{\pm}(c:d) := \partial(c:d) \pm \partial(-c:d) = \begin{bmatrix} a \\ c \end{bmatrix} - \begin{bmatrix} b \\ d \end{bmatrix} \pm \begin{bmatrix} -a \\ c \end{bmatrix} \mp \begin{bmatrix} -b \\ d \end{bmatrix},$$

extended by linearity. Then

$$(c:d) \to \{\frac{b}{d}, \frac{a}{c}\} \pm \{\frac{-b}{d}, \frac{-a}{c}\}$$

gives an isomorphism $H^{\pm}(\mathfrak{a}) \to V^{\pm}(\mathfrak{a})$. *It is understood that one choice of sign is taken consistently throughout.*

§4.4 The Algorithm for $\mathbb{Q}(\sqrt{-3})$

Let $\rho = \frac{1}{2}(1+\sqrt{-3})$ and $\omega = \rho-1 = \frac{1}{2}(-1+\sqrt{-3})$. Recall from §2.4 that $\Delta = SL(2,Z[\rho])$ is generated by $S,T,U = \begin{pmatrix}1 & \rho\\ 0 & 1\end{pmatrix}$ and $R = \begin{pmatrix}0 & \rho\\ \omega & 0\end{pmatrix}$. It will be convenient to consider, as well as Δ, the larger group $\Delta' = \{\begin{pmatrix}a & b\\ c & d\end{pmatrix} : a,b,c,d \in Z[\rho],\ ad-bc = \pm 1\}$ so that $\Delta' = \Delta \cup \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\Delta$. Note that the projectivization $\overline{\Delta}'$ is the whole of $PGL(2,Z[\rho])$ since the group of units modulo squares, $O_K^*/(O_K^*)^2$, has order 2 and $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ lies in the nontrivial coset. A fundamental region for Δ' is given by cutting the region D of §2.4 in half; if we then replace half of this by its image under S, we obtain a fundamental region F for Δ' with vertices at 0, ∞, $P_1 = ((1+\rho)/3, \sqrt{(2/3)})$ and $P_2 = (\frac{1}{2},0,\frac{1}{2}\sqrt{3})$ (see Diagram 4.4). Let $P_3 = ((2\rho-1)/3, \sqrt{(2/3)})$. Then the edge P_1P_2 is stabilized as usual by $\langle TS\rangle$, of order 3, while edge P_2P_3 is stabilized by $\langle V\rangle$ where $V = TU^{-1}RSR$, also of order 3. Vertex P_2 has stabilizer $G_P = \langle TS,V\rangle$ which is of order 12, and is contained in Δ since $\det V = \det TS = 1$. The 12 transforms of F under G_P thus all have a vertex at P_2; they fill out a tetrahedron F+ with vertices at 0, 1, ∞ and ρ (see Diagram 4.4). The six edges of F+ consist of three vertical half-lines from ∞ to 0, 1, and ρ, and three semicircles in vertical planes joining 0, 1 and ρ. Each edge belongs to precisely two of the unit cells (transforms of F) and is the transform of $\{0,\infty\}$ under the corresponding two elements of G_P. This redundancy of edges will be reflected in certain extra relations among the symbols later.

So to generate the homology of $G\backslash H_3^*$ where G is a subgroup of finite index in Δ we require a symbol (γ) for every matrix γ in a set of coset representatives for G in Δ', representing the edge $\{\gamma(0),\gamma(\infty)\}$. The redundancy observed immediately above is recorded in the relation

(4.4.1) $$(\gamma) + (\gamma R) = 0$$

since R interchanges 0 and ∞. There are two types of 'face relation', namely

(4.4.2) $$(\gamma) + (\gamma TS) + (\gamma (TS)^2) = 0,$$ and

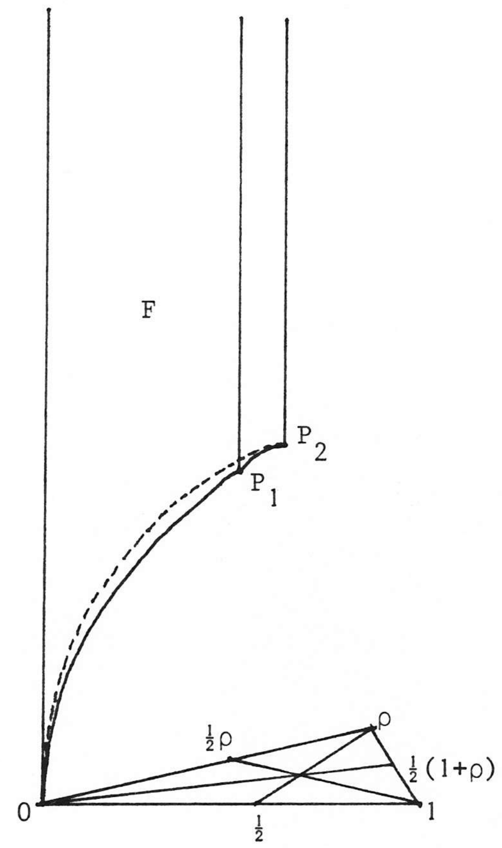
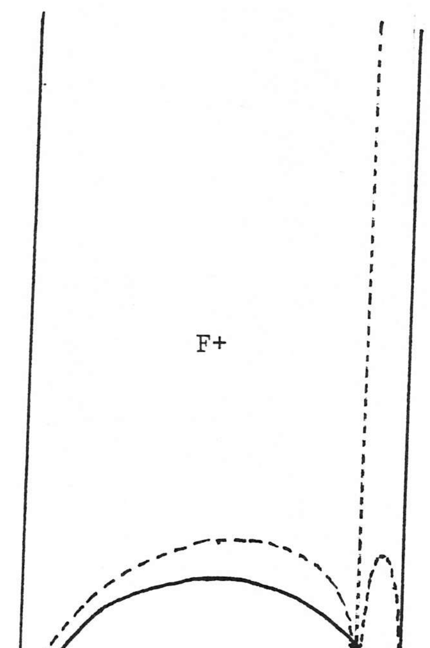
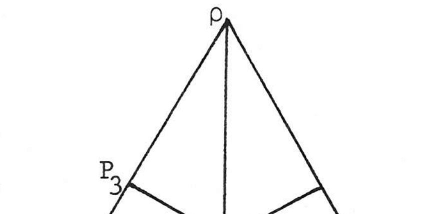


Diagram 4.4



Plan:



(4.4.3) $$(\gamma) + (\gamma V) + (\gamma V^2) = 0.$$

The edge relations record which transforms of F meet at the edge $\{0,\infty\}$, and are generated by (4.4.1) and

(4.4.4) $$(\gamma) = (\gamma I_\rho)$$

where $I_\rho = \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}$.

Now if $\{\gamma_i\}_{i=1}^k$ is a set of coset representatives for $\overline{G}$ in $\overline{\Delta}$, we can obtain a set of representatives for $\overline{G}$ in $\overline{\Delta}'$ by taking $\{\gamma_i\}_{i=1}^k \cup \{I_\rho^{-1}\gamma_i\}_{i=1}^k$. Write γ^+ for γ and γ^- for $I_\rho^{-1}\gamma$; then (4.4.1) - (4.4.4) become

(i) $$(\gamma^\pm) + (\gamma^\pm R) = 0;$$

(ii) $$(\gamma^\pm) + (\gamma^\pm TS) + (\gamma^\pm (TS)^2) = 0;$$

(iii) $$(\gamma^\pm) + (\gamma^\pm V) + (\gamma^\pm V^2) = 0;$$

(iv) $$(\gamma^\pm) = (\gamma^\pm I_\rho).$$

The involution J induced by conjugation with I_ρ can be written

$$(\gamma) \to (I_\rho^{-1}\gamma I_\rho)$$

so we have $$J(\gamma^+) = (\gamma^- I_\rho)$$

and $$J(\gamma^-) = (\gamma^+ I_\rho);$$

using (iv) we get

$$J(\gamma^+) = (\gamma^-) \quad ; \quad J(\gamma^-) = (\gamma^+).$$

Also, $I_\rho^{-1}TSI_\rho = V$, so that each relation of type (iii) is obtained by applying J to a relation of type (ii). Hence if we impose the extra relations

$$(\gamma) = J(\gamma)$$

or $$(\gamma) = -J(\gamma)$$

we can dispense with the (γ^-) symbols and with relations (iii) and (iv). We can also replace (i) with

$$(\gamma) + (\gamma S) = 0$$

since $RS = I_\rho^2$ which by (iv) acts trivially. So much simplification is gained by calculating the eigenspaces for J separately, and we have the following result.

<u>Theorem 4.4.5</u> *Form the* $\mathbb{Q}$*-vector space with symbols* (γ) *as basis, for a complete set of coset representatives* γ *of* G *in* Δ, *modulo all relations*

of the form

(i) $$(\gamma) = \pm(\gamma I_\rho);$$

(ii) $$(\gamma) + (\gamma S) = 0;$$

(iii) $$(\gamma) + (\gamma TS) + (\gamma(TS)^2) = 0.$$

Let $H^{\pm}(G)$ *denote the kernel of* $\partial^{\pm} = \partial \pm \partial J$. *Then* $H^{\pm}(G)$ *is isomorphic to the eigenspace of* $H_1(G\backslash H_3^*,\mathbb{Q})$ *on which* J *acts as* ± 1.

Of course when $G = \Delta_0(\mathfrak{a})$ we can use M-symbols as coset representatives for G in Δ. Relations (4.4.5)(i) - (iii) become

(i) $$(c:d) = \pm(\rho c:d);$$

(ii) $$(c:d) + (-d:c) = 0;$$

(iii) $$(c:d) + (c+d:-c) + (d:-c-d) = 0.$$

The adjusted boundary operators $\partial^{\pm}$ have formulae

$$\partial^{\pm}((c:d)) = \left[\frac{a}{c}\right] - \left[\frac{b}{d}\right] \pm \left(\left[\frac{\rho a}{c}\right] - \left[\frac{\rho b}{d}\right]\right).$$

§4.5 The Algorithm for $Q(\sqrt{-2})$

Let $\theta = \sqrt{-2}$ and $\Delta = SL(2,Z[\theta])$, and set $U = \begin{pmatrix}1 & \theta\\ 0 & 1\end{pmatrix}$. Let $\Delta' = \Delta \cup \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\Delta$ be the larger group of matrices with determinant ± 1. It follows from §2.4 that a fundamental region for the action of Δ' on H_3^* is the spike-shaped region F with vertices at 0, ∞, $P_1 = (\frac{1}{2},\frac{1}{2}\sqrt{3})$, $P_2 = (\frac{1}{2}(1+\theta),\frac{1}{2})$, and $P_3 = (\frac{1}{2}\theta,\frac{1}{2}\sqrt{2})$. As usual, edge P_1P_2 is stabilized by $\langle TS\rangle$, of order 3. Now P_2P_3 is stabilized by a group of order 4 generated by USI_{-1} (which has determinant -1). The full stabilizer of P_2 is thus $\langle TS,USI_{-1}\rangle$ which has order 24. The 24 transforms of F under this group G_P each have a vertex at P_2 and fit together around P_2 to form the basic polyhedron F+, which in this case is a cuboctahedron (with six quadrilateral faces and eight triangular faces). For a picture of F+ and a projection onto the floor, see Diagram 4.5. The 24 edges are the images of $\{0,\infty\}$ under G_P, precisely one to each transform of F. So as in the case of $Q(\sqrt{-3})$, to generate the 1-homology of $G\backslash H_3^*$ for a subgroup G of Δ or Δ' we must use symbols (γ) for each matrix γ in a complete set of coset representatives for G in Δ', corresponding to the edge $\{\gamma(0),\gamma(\infty)\}$. As face relations we have: for the triangles,

(4.5.1) $$(\gamma) + (\gamma TS) + (\gamma(TS)^2) = 0,$$

and for the quadrilaterals,

(4.5.2) $$(\gamma) + (\gamma USI_{-1}) + (\gamma(USI_{-1})^2) + (\gamma(USI_{-1})^3) = 0.$$

As for the edge relations: four copies of F+ meet at $\{0,\infty\}$, namely its images under I, I_{-1}, S and SI_{-1}, giving us the relations

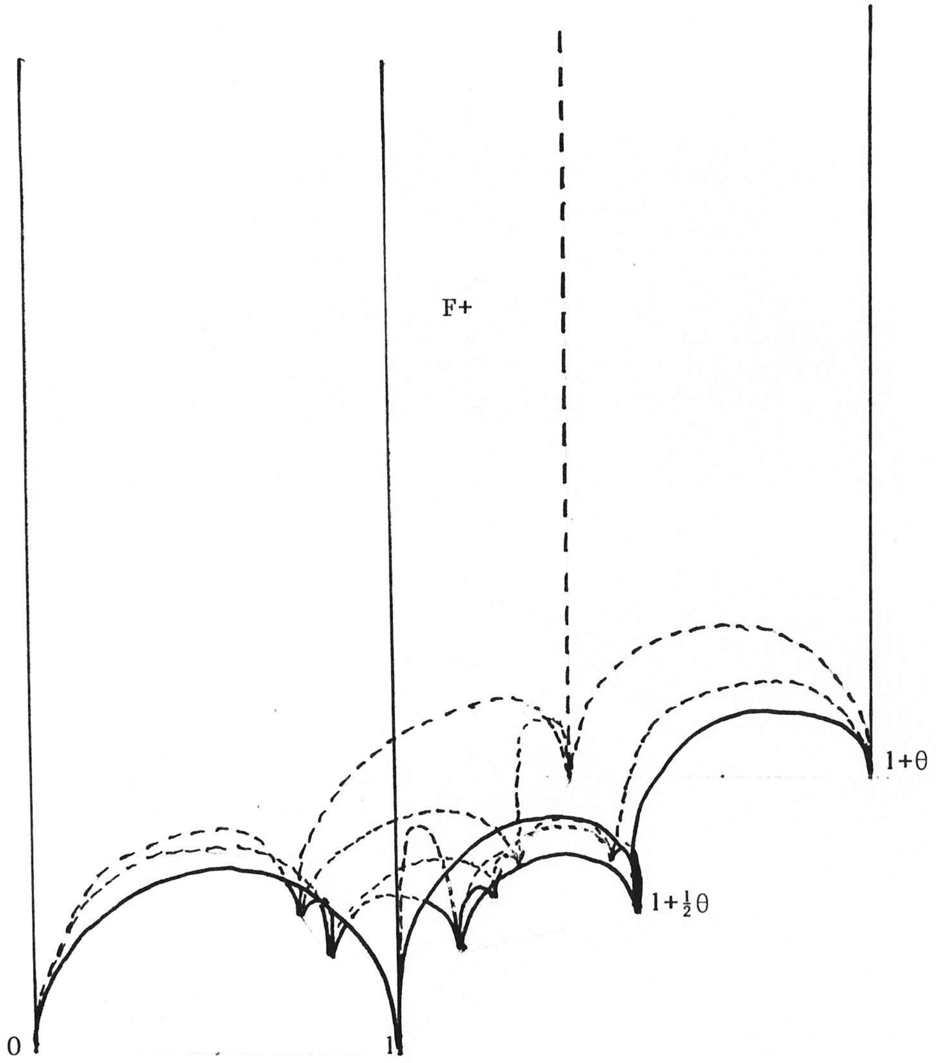
(4.5.3) $$(\gamma) + (\gamma S) = 0;$$

(4.5.4) $$(\gamma) = (\gamma I_{-1}).$$

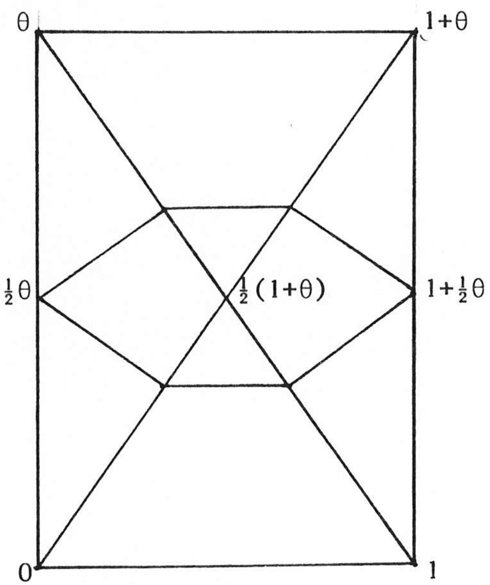
As with $Q(\sqrt{-3})$ we can convert the symbols and relations in such a way as to involve only a set of coset representatives for G in Δ. Let $\{\gamma\}$ be such a set, and let $\gamma^+ = \gamma$ and $\gamma^- = I_{-1}\gamma$, so that $\{\gamma^+\} \cup \{\gamma^-\}$ is a set of coset representatives for G in Δ'. Then relations (4.5.1),(4.5.3) become

$$(\gamma^{\pm}) + (\gamma^{\pm}TS) + (\gamma^{\pm}(TS)^2) = 0$$

and $$(\gamma^{\pm}) + (\gamma^{\pm}S) = 0,$$



Plan:



The vertices not labelled are at $(1+\theta)/3$, $(2+\theta)/3$, $(1+2\theta)/3$ and $(2+2\theta)/3$.

Diagram 4.5

or $$(\gamma^{\pm}) \;+\; ((\gamma TS)^{\pm}) \;+\; ((\gamma(TS)^2)^{\pm}) \;=\; 0$$

and $$(\gamma^{\pm}) \;+\; ((\gamma S)^{\pm}) \;=\; 0;$$

while because I_{-1} has determinant -1 we have $(\gamma^{\pm}I_{-1}) = (I_{-1}\gamma^{\mp}I_{-1}) = J(\gamma^{\mp})$; hence (4.5.4) becomes

$$(\gamma^{\pm}) \;=\; J(\gamma^{\mp})$$

while (4.5.2) becomes

$$(\gamma) \;+\; (\gamma US) \;+\; (\gamma(USI_{-1}US)) \;+\; (\gamma(USI_{-1})^2 US) \;=\; 0.$$

As before, we can simplify considerably by assuming that J acts as a scalar ± 1; that is, by including extra relations $(\gamma) = \pm J(\gamma)$. The result is:

<u>Theorem 4.5.5</u> *Form the* $\mathbb{Q}$*-vector space with symbols* (γ) *as basis, for a complete set of coset representatives* γ *for* G *in* Δ, *modulo all relations of the form*

(i) $$(\gamma) \;=\; \pm(\gamma I_{-1});$$

(ii) $$(\gamma) \;+\; (\gamma S) \;=\; 0;$$

(iii) $$(\gamma) \;+\; (\gamma TS) \;+\; (\gamma(TS)^2) \;=\; 0;$$

(iv) $$(\gamma) \;+\; (\gamma US) \;+\; (\gamma(USI_{-1})^2) \;+\; (\gamma(USI_{-1})^2 US) \;=\; 0.$$

Let $H^{\pm}(G)$ *denote the kernel of* $\partial^{\pm} = \partial \pm \partial J$. *Then* $H^{\pm}(G)$ *is isomorphic to the eigenspace of* $H_1(G\backslash H_3^*,\mathbb{Q})$ *on which* J *acts as* ± 1.

In terms of M-symbols, when $G = \Delta_0(\mathfrak{a})$, the relations are

(i) $$(c:d) \;=\; \pm(-c:d);$$

(ii) $$(c:d) \;+\; (-d:c) \;=\; 0;$$

(iii) $$(c:d) \;+\; (c+d:-c) \;+\; (d:-c-d) \;=\; 0;$$

(iv) $$(c:d) \pm (\theta c+d:c) \;+\; (-c+\theta d:\theta c+d) \pm (d:c-\theta d) \;=\; 0;$$

the adjusted boundary maps $\partial^{\pm}$ have formulae

$$\partial^{\pm}(c:d) \;=\; \begin{bmatrix}a\\c\end{bmatrix} - \begin{bmatrix}b\\d\end{bmatrix} \pm \left(\begin{bmatrix}-a\\c\end{bmatrix} - \begin{bmatrix}-b\\d\end{bmatrix}\right).$$

The reason for the alternating signs in (iv) above is the following: for computational convenience we wish to express the terms in (iv) as an orbit of a single matrix USI_{-1}; but USI_{-1} has determinant -1; so if $ad-bc = 1$ then

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} USI_{-1} \;=\; \begin{pmatrix}a\theta+b & a\\ c\theta+d & c\end{pmatrix}$$

where the matrix on the right-hand side has determinant -1; now according to (4.2.5) the latter corresponds to the path $\{\frac{a}{c},\frac{a\theta+b}{c\theta+d}\}$. However, the symbol $(\theta c+d:c)$ corresponds to $\{\frac{x}{c},\frac{y}{c\theta+d}\}$ where $yc - x(c\theta+d) = +1$, so that we may take $x = -a$, and $y = -(a\theta+b)$, and then

$$\{\frac{a}{c},\frac{a\theta+b}{c\theta+d}\} = \{\frac{-x}{c},\frac{-y}{c\theta+d}\} = J(\{\frac{x}{c},\frac{y}{c\theta+d}\})$$
$$= \pm\{\frac{x}{c},\frac{y}{c\theta+d}\},$$

according to which sign we have chosen in (i).

§4.6 The Algorithm for $\mathbb{Q}(\sqrt{-7})$

Let $\alpha = \frac{1}{2}(1+\sqrt{-7})$ and $\Delta = SL(2,\mathbb{Z}[\alpha])$, and set $U = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. Again we will have to consider the larger group $\Delta' = \Delta \cup \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. In §2.4 we gave a fundamental region for Δ, consisting of points (z,t) such that

a) $|z|^2 + t^2 \geq 1$, so that (z,t) is outside the unit sphere;

b) z is inside the 'fundamental hexagon' of points in the complex plane nearer to 0 than to any other element of $\mathbb{Z}[\alpha]$ (see Diagram 4.6).

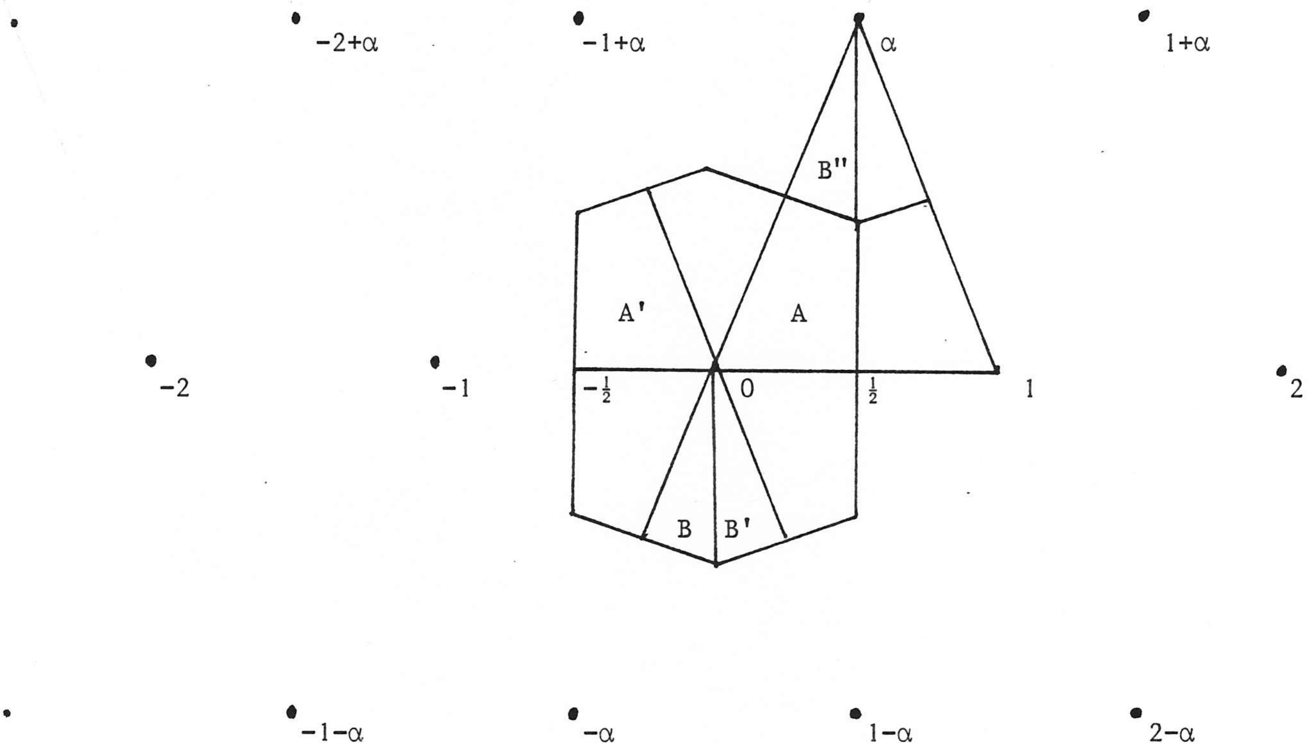


Diagram 4.6

It is clear that we may obtain a fundamental region for Δ' by keeping only those (z,t) for which z lies in one of the regions A, A', B, or B'. Replace the points above A' and B' (which are above the unit sphere) with their images under S: these lie within the unit sphere and over A and B. Lastly, translate all the points above B by U so that they lie above B'' (see diagram). This gives a fundamental region F for Δ' with vertices

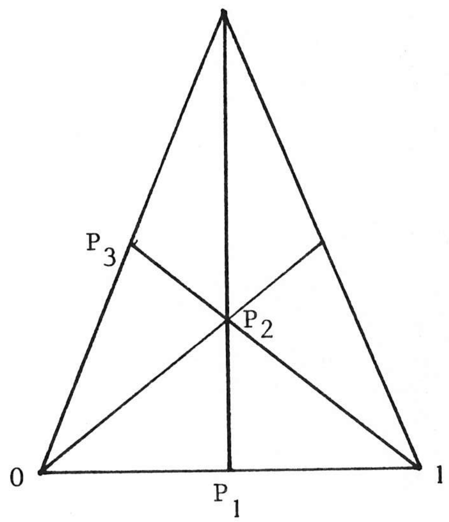


Diagram 4.7

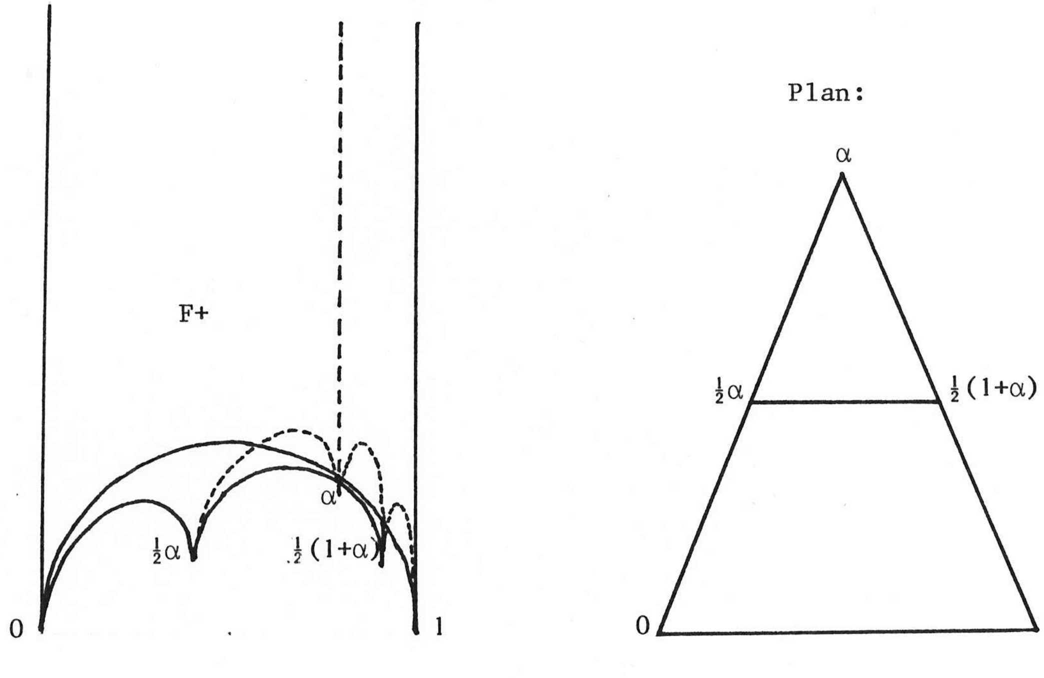


Diagram 4.8

at 0, ∞, α, $P_1 = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $P_2 = ((3\alpha+2)/7, \sqrt{(3/7)})$, and $P_3 = (\frac{1}{2}\alpha, \frac{1}{2}\sqrt{2})$: see Diagram 4.7 for a plan. The stabilizer G_P of P_2 has order 6, and is generated by TS (which has order 3 and fixes the edge P_1P_2), and $Y = \begin{pmatrix} 1 & -\alpha \\ 1-\alpha & -1 \end{pmatrix}$ (which has order 2 and fixes the edge P_2P_3). These satisfy $YTSY = (TS)^2$, and so G_P is isomorphic to the dihedral group of order 6.

As before, we let $F+ = \bigcup_{\gamma \in G_P} (\gamma F)$ be the basic polyhedron. It is a triangular prism with vertices at 0, 1, ∞, α, $\frac{1}{2}\alpha$, and $\frac{1}{2}(1+\alpha)$: see Diagram 4.8. There are nine edges: six are the images under G_P of $\{0,\infty\}$, and bound the triangular faces $\langle 0,1,\infty\rangle$ and $\langle \alpha, \frac{1}{2}\alpha, \frac{1}{2}(1+\alpha)\rangle$; the others are the images under G_P of the edge $\{\alpha,\infty\}$ of F, each occurring on precisely two of the images of F.

So to generate the 1-homology we have to start with <u>two</u> kinds of generator: the usual symbol (γ) representing $\{\gamma(0),\gamma(\infty)\}$, and a second symbol $[\gamma]$ representing $\{\gamma(\alpha),\gamma(\infty)\}$. We clearly have the relation

(4.6.1) $$[\gamma] = (\gamma U),$$

and because the edges $[\gamma]$ each lie on <u>two</u> images of F, we have the relation

(4.6.2) $$[\gamma] + [\gamma TSY] = 0.$$

As for face relations we have first, for the triangles,

(4.6.3) $$(\gamma) + (\gamma TS) + (\gamma(TS)^2) = 0,$$

and for the quadrangles:

(4.6.4) $$(\gamma) - [\gamma] + (\gamma Y) - [\gamma Y] = 0.$$

As edge relations, the only ones we need are

(4.6.5) $$(\gamma) + (\gamma S) = 0, \text{ and}$$

(4.6.6) $$(\gamma) = (\gamma I_{-1}),$$

since the other is $(\gamma) + (\gamma SI_{-1}) = 0$, which is a consequence of these; also, (4.6.2) is now redundant, being a consequence of (4.6.1) and (4.6.5) since we have the matrix identity $TSY = USU^{-1}$. So we may ignore the second type of symbol altogether , if we use relations

(4.6.7)(i) $$(\gamma) = (\gamma I_{-1});$$

(ii) $$(\gamma) + (\gamma S) = 0;$$

(iii) $(\gamma) + (\gamma TS) + (\gamma(TS)^2) = 0;$

(iv) $(\gamma) - (\gamma U) + (\gamma Y) - (\gamma YU) = 0.$

Theorem 4.6.8 *Form the* $\mathbb{Q}$*-vector space with symbols* (γ) *as basis, for a complete set of coset representatives* γ *for* G *in* Δ'*, modulo all relations of the form (4.6.7)(i), (ii), (iii), and (iv). Let* $H(G)$ *denote the kernel of the boundary map* ∂*, defined by (4.2.1) and extended by linearity. Then (4.2.5) gives an isomorphism from* $H(G)$ *to* $H_1(G\backslash\mathcal{H}_3^*,\mathbb{Q})$.

As before we can make do with the smaller set of symbols corresponding to a set of coset representatives for G in Δ by adjusting the relations suitably. We omit the details, which are simpler than for $\mathbb{Q}(\sqrt{-2})$ since all the matrices appearing in relations (4.6.7) have determinant +1 (except for I_{-1}), and merely state the result in terms of M-symbols for $G = \Delta_0(\mathfrak{a})$.

Theorem 4.6.9 *Form the* $\mathbb{Q}$*-vector space with symbols* $(c:d)$ *as basis, for each* $(c:d) \in P^1(\mathfrak{a})$*, modulo all relations of the form*

(i) $(c:d) = \pm(-c:d);$

(ii) $(c:d) + (-d:c) = 0;$

(iii) $(c:d) + (c+d:-c) + (d:-c-d) = 0;$

(iv) $(c:d) + (\alpha c+d:-c) + (c+(1-\alpha)d:-\alpha c-d) + (-d:c+(1-\alpha)d) = 0.$

Let $H^{\pm}(\mathfrak{a})$ *be the kernel of* $\partial^{\pm} = \partial \pm \partial J$*. Then* $H^{\pm}(\mathfrak{a})$ *is isomorphic to* $V^{\pm}(\mathfrak{a})$.

§4.7 The Algorithm for $\mathbb{Q}(\sqrt{-11})$

Here the geometric situation is very similar to that for $\mathbb{Q}(\sqrt{-7})$: we may make use of Diagrams 4.6 and 4.7 and construct the fundamental region F in a similar manner. Now, of course, $\alpha = \frac{1}{2}(1+\sqrt{-11})$, and the vertex P_2 of F has coordinates $((3+5\alpha)/11, \sqrt{(2/11)})$. The stabilizer G_P of P_2 has order 12: it is generated by TS, of order 3, and $X = \begin{pmatrix} -1 & \alpha \\ \alpha-1 & 2 \end{pmatrix}$, which also has order 3; they satisfy $(XTS)^2 = I$. The fundamental polyhedron $F+ = \bigcup_{\gamma \in G_P} (\gamma F)$ is now a truncated tetrahedron, with four triangular and four hexagonal faces: see Diagram 4.9.

We again let $(\gamma) = \{\gamma(0), \gamma(\infty)\}$ and $[\gamma] = \{\gamma(\alpha), \gamma(\infty)\}$. The triangular faces of F+ have edges (γ) for $\gamma \in G_P$; the hexagonal faces have alternate edges of type (γ) and $[\gamma]$. So the relations are:

(4.7.1)(i) $[\gamma] = (\gamma U)$;

(ii) $[\gamma] + [\gamma TSX] = 0$;

(iii) $(\gamma) + (\gamma S) = 0$;

(iv) $[\gamma] = [\gamma U I_{-1} U^{-1}]$;

(v) $(\gamma) = (\gamma I_{-1})$;

(vi) $(\gamma) + (\gamma TS) + (\gamma (TS)^2) = 0$;

(vii) $(\gamma) - [\gamma] + (\gamma X) - [\gamma X] + (\gamma X^2) - [\gamma X^2] = 0$.

Of these, (ii) and (iv) follow from the others, and by means of (i) we can avoid the use of the second type of symbol altogether.

Theorem 4.7.2 *Form the* $\mathbb{Q}$*-vector space with symbols* (γ) *as basis, for a complete set of coset representatives* γ *for* G *in* Δ'*, modulo all relations of the form*

(i) $(\gamma) = (\gamma I_{-1})$;

(ii) $(\gamma) + (\gamma S) = 0$;

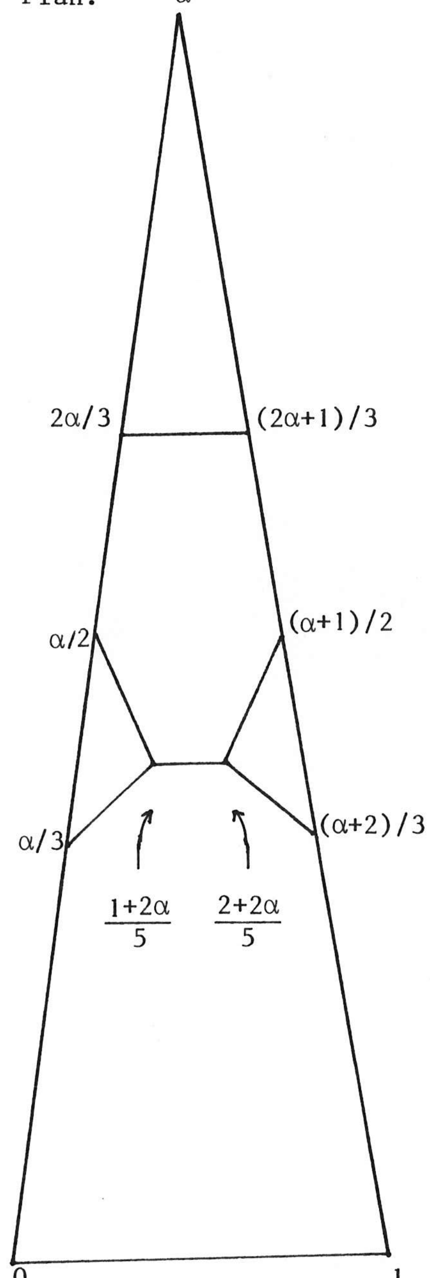
(iii) $(\gamma) + (\gamma TS) + (\gamma (TS)^2) = 0$;

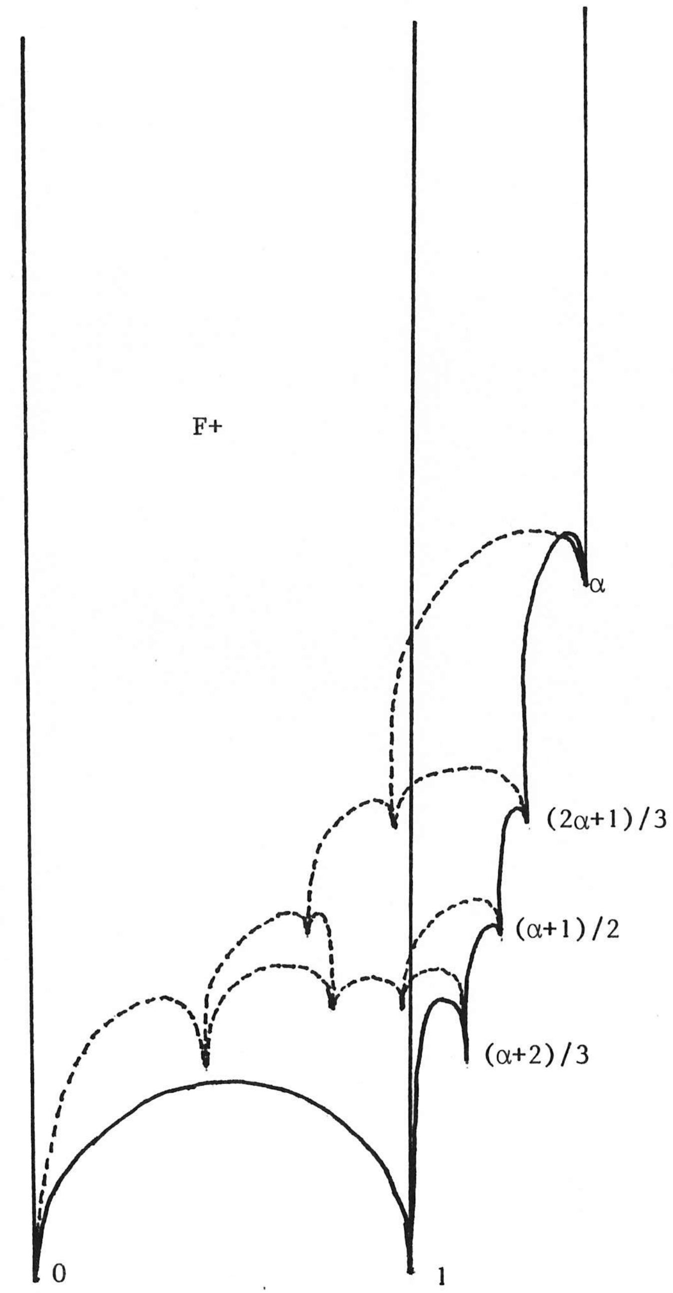
(iv) $(\gamma) - (\gamma U) + (\gamma X) - (\gamma XU) + (\gamma X^2) - (\gamma X^2 U) = 0$.

Let H(G) *denote the kernel of* ∂. *Then* H(G) *is isomorphic to* $H_1(G\backslash H_3^*, \mathbb{Q})$ *via (4.2.5).*

Diagram 4.9

Plan:





In terms of M-symbols for $\Delta_0(a)$, the result reads as follows.

Theorem 4.7.3 *Form the* $\mathbb{Q}$*-vector space with symbols* $(c:d)$ *as basis, for each* $(c:d) \in P^1(a)$, *modulo all relations of the form*

(i) $$(c:d) = \pm(-c:d);$$

(ii) $$(c:d) + (-d:c) = 0;$$

(iii) $$(c:d) + (c+d:-c) + (d:-c-d) = 0;$$

(iv) $$(c:d) + (\alpha c+d:-c) + (-c+(\alpha-1)d:c\alpha+2d) + (-d:c+(1-\alpha)d) + (-2c+(\alpha-1)d:\alpha c+d) + (\alpha c+2d:-2c+(\alpha-1)d) = 0.$$

Let $H^{\pm}(a)$ *denote the kernel of* $\partial^{\pm} = \partial \pm \partial J$. *Then* $H^{\pm}(a)$ *is isomorphic to* $V^{\pm}(a)$.

§4.8 Other Quadratic Fields

In order to extend the algorithms just described for the Euclidean fields to the other complex quadratic fields K with unique factorization (namely $\mathbb{Q}(\sqrt{-d})$ for d = 19, 43, 67 and 163), one encounters some geometric and some algebraic difficulties, but these do not seem insurmountable. Because not every element of K has a representative modulo 0_K of norm less than 1, the fundamental region for $SL(2,0_K)$ acting on H_3^* has a curved floor consisting of more than one section of sphere (whereas above, the unit sphere with centre 0 sufficed): for example, see Swan [20] section 16, where the case $\mathbb{Q}(\sqrt{-19})$ is worked out. However, similar arguments to the ones used above would probably produce suitable 'fundamental polyhedra', leading to a result similar to Theorem 4.3.2. The other difficulty is that, of course, we would no longer have the Euclidean Algorithm; apart from the use we make of this in general arithmetic computations, we also used it in converting any modular symbol whose end-points are cusps to a sum of the form $\Sigma\{\gamma(0),\gamma(j\infty)\}$ for some matrices $\gamma \in SL(2,0_K)$. However, careful inspection of the procedure which brings an arbitrary point of H_3 to within the fundamental region (using, as well as translations, and inversions in the unit sphere with centre 0, inversions in the other unit spheres bounding the fundamental region) should yield an algorithm for these fields.

When the class number h of K is greater than one, the fundamental region has more than one cusp (as remarked in §3.2, the number of cusps is equal to the class number). This makes the geometry more complicated still, and the definition of a cusp form would need to take into account the fact that not every cusp is equivalent to the cusp at infinity under the action of $SL(2,0_K)$. Alternatively, it is quite likely that one should use not $PSL(2,0_K)$ but some larger group, under the action of which every cusp is equivalent to $j\infty$, as the main discrete group acting on H_3.

As for modular symbols, one might have to use more symbols than just those of the form $\{\gamma(0),\gamma(j\infty)\}$ to generate homology; details of the calculations, such as manipulation of the M-symbols, would also be more difficult. However I see no fundamental reason why the algorithms given in this chapter for the Euclidean fields could not, with more work, be extended to any complex quadratic field.

CHAPTER 5

The Computations and Results

For each of the five Euclidean fields discussed in the previous Chapter, computer programs have been written in Algol 68 which carry out the algorithms presented there, in terms of M-symbols. These programs have been run on an ICL 2980 computer at the Oxford University Computing Service. Thus we have been able to calculate, for each field K and each ideal $\mathfrak{a}$ of O_K such that $N\mathfrak{a}$ is not too large, the dimensions of $V(\mathfrak{a})$, $V^+(\mathfrak{a})$, and $V^-(\mathfrak{a})$; the actions of the main involution J, the W_π involutions for each prime π dividing $\mathfrak{a}$; the action of the Hecke operator T_π for any prime π not dividing $\mathfrak{a}$; the splitting of $V(\mathfrak{a})$ into one-dimensional spaces which are eigenspaces for all these operators; and the eigenvalues on each such subspace. By inspection, we can easily determine which of these eigenspaces correspond to oldforms, since we will have already met them as newforms for $\Delta_0(\mathfrak{b})$ for some $\mathfrak{b}$ dividing $\mathfrak{a}$.

For each field we give first a table showing for each ideal $\mathfrak{a}$ (with norm up to some bound) the dimensions of $V(\mathfrak{a})$, $V^+(\mathfrak{a})$, and $V^-(\mathfrak{a})$ as well as the dimensions of the corresponding spaces of newforms for $\Delta_0(\mathfrak{a})$. Then for the "+" and "−" spaces separately we list, for each level, the first fifteen Hecke eigenvalues for each newform. Thus two limits had to be set for each field: the upper bounds for the norm of the level $\mathfrak{a}$, and for the number of Hecke operators of which to calculate the action. These limits were decided in terms of how much computer time was available: in all cases the physical limitations (storage space and size of integers encountered) would have allowed the computations to be extended much further. For example, for $\mathbb{Q}(i)$, the systematic coverage of all levels stops at norm 500, but a few isolated levels were calculated up to $(1+i)^{12}$, with norm 4096.

As well as the systematic coverage of all levels $\mathfrak{a}$ with norm $N\mathfrak{a}$ less than a certain bound, a few sporadic cases were also computed for ideals of larger norm: firstly, to gather evidence for Claim B of §5.6 before this was proved, and secondly when it was known that there existed elliptic curves with the corresponding conductor. For example, R.J.Stroeker's thesis [19] gives tables of all elliptic curves over $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$ with bad reduction only at the prime dividing 2; for $\mathbb{Q}(\sqrt{-1})$, extra calculations were done with powers of $(1+i)$.

For the first three fields, some of these calculations have been previously carried out by Mennicke and Grunewald, working in Bielefeld. They only work with split prime ideals, for which the M-symbols just reduce to elements of the projective line over a finite field $GF(p)$, for a rational prime p. The relations they use are derived in an algebraic, rather than a geometric way, described in [11] for the case $\mathbb{Q}(\sqrt{-1})$. In this paper they also give results for $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{-3})$, which agree with the tables in this Chapter insofar as they overlap.

We have also made a systematic search for elliptic curves with small conductor over each of the five fields. Here we implement on the computer Tate's algorithm (c.f.[2]) to determine the type of the reduction of an elliptic curve at a prime $\mathfrak{p}$, given its coefficients a_1,a_2,a_3,a_4 and a_6 (see equation (1.4.1)). It is easily seen from the formulae given by Tate (op. cit.) that we may assume that a_1, a_2, and a_3 are reduced modulo 2, 3, and 2 respectively; so the search consists of a systematic stepping through an enumeration of the pairs (a_4,a_6): for each pair, all values of a_1 and a_3 (mod 2) and a_2 (mod 3) are considered.

In the next five sections the following tables are given for each field:

1) The dimensions of $V(\mathfrak{a})$, $V^+(\mathfrak{a})$, and $V^-(\mathfrak{a})$, and the corresponding subspaces of newforms, for each ideal $\mathfrak{a}$, with norm less than a fixed bound. This bound was 500, 300, 500, 200, or 200 for $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-7})$, and $\mathbb{Q}(\sqrt{-11})$ respectively. Levels $\mathfrak{a}$ with $\dim V(\mathfrak{a}) = 0$ are omitted

from the Table for brevity. Also, only one ideal of each conjugate pair a, $\overline{a}$ is given, since obviously conjugation induces an isomorphism from $V(a)$ to $V(\overline{a})$.

2) A list of the first fifteen Hecke eigenvalues for each 'newform' in $V^+(a)$, together with the eigenvalues of the W involutions.

3) As for Table 2, but for $V^-(a)$.

4) A list of elliptic curves defined over K, in order of the norm of the conductor. The coefficients and various invariants of each curve are given, as well as the Trace of Frobenius at each of the first fifteen primes (in a separate Table). Only one of each pair of conjugate curves is listed; only one curve from each isomorphism class is included; isogenies between listed curves are indicated.

In Table 4, no claim of completeness is made. There are, almost certainly, more curves with small conductor, with coefficients outside the search region. In particular, we expect there to exist an isogeny class of curves with conductor a to correspond to each newform in $V^+(a)$ (see §5.6), but comparison of Tables 2 and 4 for each field will reveal that some of these expected curves have not yet been found. There is some precedent for this in the rational case: for some conductors, for example 78, no curves were found by any systematic search. Of course Tingley's method, described earlier, gives a method for constructing curves directly from newforms, in the rational case. Eventually we expect to fill the gaps in the Tables for the five Euclidean fields, but at present we give, at the foot of Table 4 for each field, a list of 'missing' conductors.

For the field $\mathbf{Q}(\sqrt{-1})$, extra tables are given of curves with conductor a power of $(1+i)$: these are taken from [19]; there are also extra tables of newforms at the corresponding levels.

§5.1 The Results of Computations for $\mathbb{Q}(\sqrt{-1})$

Table 5.1.1: Ideals a of $Z[i]$ with $Na \leq 500$ and $\dim V(a) > 0$

Table 5.1.1a: $\operatorname{Dim} V((1+i)^e)$ for $e \leq 12$

Table 5.1.2: Rational Newforms in $V^+(a)$

Table 5.1.3: Rational Newforms in $V^-(a)$

Table 5.1.4: Elliptic Curves with small Conductor

Table 5.1.4a: Elliptic Curves with Conductor $(1+i)^e$ for $e \leq 12$

Table 5.1.5: Elliptic Curves with small Conductor: Zeta Functions

Table 5.1.5a: Elliptic Curves with Conductor $(1+i)^e$: Zeta Functions

Table 5.1.1: Ideals a of Z[i] with Na ≤ 500 and dim V(a) > 0

Only one ideal of each conjugate pair is given. Numbers in parentheses refer to dimensions of spaces of newforms. The 'splitting field', which is Q unless otherwise specified, is the smallest extension of Q containing all the eigenvalues of the Hecke algebra acting on V(a).

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
(7 + 4i)	65	1	1	0	
(8 + 2i)	68	1	0	1	
(6 + 6i)	72	1	1	0	
(7 + 7i)	98	1	1	0	
(10)	100	1	1	0	
(9 + 5i)	106	1	1	0	
(11)	121	1	1	0	
(9 + 7i)	130	1	1	0	
(11 + 3i)	130	2 (0)	2 (0)	0	
(10 + 6i)	136	2 (0)	0	2 (0)	
(11 + 4i)	137	1	0	1	
(12)	144	3 (1)	2 (0)	1	
(9 + 8i)	145	2	0	2	
(12 + i)	145	1	0	1	
(12 + 4i)	160	2	1	1	
(10 + 8i)	164	1	1	0	
(12 + 6i)	180	1	0	1	
(13 + 4i)	185	1	0	1	
(13 + 5i)	194	2	2	0	
(14)	196	2 (0)	2 (0)	0	
(10 + 10i)	200	3 (1)	3 (1)	0	
(14 + 4i)	212	2 (0)	2 (0)	0	
(15)	225	1	1	0	
(13 + 8i)	233	1	1	0	
(11 + 11i)	242	2 (0)	2 (0)	0	
(16 + i)	257	1	1	0	
(14 + 8i)	260	3 (0)	3 (0)	0	
(16 + 2i)	260	2 (0)	2 (0)	0	
(16 + 3i)	265	1	1	0	
(16 + 4i)	272	4 (1)	1	3 (0)	
(15 + 7i)	274	2 (0)	0	2 (0)	

Table 5.1.1 (Continued)

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
(14 + 9i)	277	1	1	0	
(12 + 12i)	288	5 (0)	3 (0)	2 (0)	
(17)	289	1	1	0	
(17 + i)	290	5 (1)	1	4 (0)	
(13 + 11i)	290	2 (0)	0	2 (0)	
(15 + 9i)	306	1	0	1	
(17 + 5i)	314	1	1	0	
(16 + 8i)	320	4 (0)	2 (0)	2 (0)	
(18)	324	1	1	0	
(15 + 10i)	325	3 (1)	2 (0)	1	
(17 + 6i)	325	1	0	1	
(18 + i)	325	3 (1)	2 (0)	1	
(18 + 2i)	328	3 (1)	2 (0)	1	
(13 + 13i)	338	2	2	0	
(18 + 4i)	340	2 (0)	0	2 (0)	
(14 + 12i)	340	3 (1)	1	2 (0)	
(18 + 6i)	360	4 (0)	2 (0)	2 (0)	
(19)	361	3	1	2	
(19 + i)	362	2	1	1	
(15 + 12i)	369	1	0	1	
(17 + 9i)	370	3 (1)	0	3 (1)	
(16 + 11i)	377	4	3	1	$Q(\sqrt{2})$
(18 + 8i)	388	4 (0)	4 (0)	0	
(14 + 14i)	392	5 (2)	5 (2)	0	
(15 + 13i)	394	1	0	1	
(20)	400	7 (2)	5 (0)	2	
(17 + 11i)	410	2	2	0	
(19 + 7i)	410	2	1	1	
(18 + 10i)	424	4 (1)	3 (0)	1	
(19 + 8i)	425	2	1	1	
(17 + 12i)	433	2	0	2	$Q(\sqrt{2})$
(21)	441	1	1	0	
(19 + 9i)	442	2	1	1	
(18 + 11i)	445	2	0	2	$Q(\sqrt{3})$
(15 + 15i)	450	3 (1)	3 (1)	0	

Table 5.1.1 (Concluded)

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
(21 + 3i)	450	2	1	1	
(16 + 14i)	452	1	0	1	
(17 + 13i)	458	1	0	1	
(21 + 5i)	466	2 (0)	2 (0)	0	
(18 + 12i)	468	1	0	1	
(22)	484	4 (1)	4 (1)	0	
(17 + 14i)	485	2	2	0	$Q(\sqrt{2})$
(21 + 7i)	490	2 (0)	2 (0)	0	
(18 + 13i)	493	1	0	1	
(20 + 10i)	500	3 (1)	2 (0)	1	

Table 5.1.1a Dim $V((1+i)^e)$ for $e \leq 12$

This gives the same information as Table 5.1.1, for $a = (1+i)^e$. Beyond $e = 12$ the spaces V(a) could not be calculated for reasons of computer time and storage space.

e	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$
8		0	0	0
9	512	4	2	2
10	1024	12 (4)	6 (2)	6 (2)
11	2048	20 (0)	10 (0)	10 (0)
12	4096	32 (4)	16 (2)	16 (2)

Table 5.1.2 Rational Newforms in $V^{+}(a)$

For each prime π with $N_{\pi} \le 50$ we give either the eigenvalue of T_{π} (if π does not divide a) or that of W_{π} (if π divides a); the latter are denoted simply "+" or "–".

a	Na	1+i	2+i	2-i	3	3+2i	3-2i	1+4i	1-4i	5+2i	5-2i	1+6i	1-6i	5+4i	5-4i	7	Other W
(7+4i)	65	0	+	0	-2	-4	–	6	0	-6	6	2	2	0	6	-4	
(6+6i)	72	+	-2	-2	–	-2	-2	2	2	6	6	6	6	-6	-6	-14	
(7+7i)	98	+	0	0	-2	-4	-4	6	6	-6	-6	2	2	6	6	–	
(10)	100	–	+	+	-2	2	2	-6	-6	6	6	2	2	6	6	-10	
(9+5i)	106	–	-3	-3	1	-4	5	0	0	-6	3	2	2	6	-3	5	+ (7-2i)
(11)	121	-2	1	1	-5	4	4	-2	-2	0	0	3	3	-8	-8	-10	– (11)
(9+7i)	130	+	0	+	4	-4	–	-6	0	-6	0	2	2	-6	6	14	
(12+4i)	160	+	–	-2	2	-2	-2	2	-6	-2	-2	-2	6	-6	10	2	
(10+8i)	164	–	0	0	-2	2	-4	6	-6	6	0	8	-4	+	-6	2	
(13+5i)	194	+	3	-3	4	-4	-4	3	0	6	-3	-7	-7	0	-9	-4	– (9-4i)
(13+5i)	194	–	-1	-1	-4	0	0	-3	4	2	9	3	-11	0	7	8	+ (9-4i)
(10+10i)	200	–	–	–	-6	-2	-2	2	2	-2	-2	6	6	-6	-6	2	
(15)	225	-1	–	–	–	-2	-2	2	2	-2	-2	-10	-10	10	10	-14	
(13+8i)	233	-2	-3	-2	-4	3	-3	-4	-3	6	-7	4	4	12	-5	1	+ (13+8i)
(16+i)	257	-1	-2	2	2	6	-2	2	2	-6	-6	-10	-6	10	2	2	– (16+i)
(16+3i)	265	0	0	–	4	2	2	-6	-6	6	-6	2	-4	0	-6	2	+ (7-2i)
(16+4i)	272	+	2	-2	2	-6	2	2	–	-6	-2	2	-2	2	-6	2	
(14+9i)	277	-1	-4	-1	-1	-4	-6	-1	2	1	5	-6	-1	6	6	-1	+ (14+9i)
(17)	289	-1	-2	-2	-6	-2	-2	–	–	6	6	-2	-2	-6	-6	2	
(17+i)	290	–	0	+	-2	-4	2	0	0	+	0	-10	8	-6	6	-10	
(17+5i)	314	+	-4	-3	-1	-3	-1	-7	5	2	-6	-2	-2	-5	4	4	+ (11-6i)
(18)	324	–	0	0	+	2	2	0	0	0	0	-10	-10	0	0	2	

Table 5.1.2 (Concluded)

a	Na	1+i	2+i	2-i	3	3+2i	3-2i	1+4i	1-4i	5+2i	5-2i	1+6i	1-6i	5+4i	5-4i	7	Other W
(13+13i)	338	+	-3	-3	-5	-	-	-3	-3	6	6	-7	-7	0	0	-13	
(13+13i)	338	-	-1	-1	3	+	+	-3	-3	2	2	3	3	0	0	-13	
(14+12i)	340	-	0	+	4	2	-4	6	+	-6	-6	-10	2	0	-6	-4	
(19)	361	0	3	3	-2	-4	-4	-3	-3	6	6	2	2	-6	-6	-13	- (19)
(19+i)	362	+	0	3	-2	5	-4	-6	3	-9	6	2	-7	6	9	-4	- (9+10i)
(16+11i)	377	0	3	0	-2	5	-	-3	0	3	+	-7	-7	-9	-3	-4	
(14+14i)	392	-	-4	-4	-2	0	0	-2	-2	2	2	-6	-6	-2	-2	-	
(14+14i)	392	+	2	2	-6	2	2	-6	-6	6	6	-2	-2	2	2	-	
(17+11i)	410	+	0	-	4	-4	2	0	-6	6	6	2	-4	6	+	-10	
(17+11i)	410	+	-4	+	-4	0	-6	0	2	-6	-2	-2	8	6	+	-2	
(19+7i)	410	-	-2	-	2	-2	-2	2	-6	-2	6	-10	-2	-	10	2	
(19+8i)	425	2	0	-	-4	0	2	+	0	6	2	-4	6	8	-4	-4	
(21)	441	-1	-2	-2	-	-2	-2	-6	-6	-2	-2	6	6	2	2	-	
(19+9i)	442	+	0	3	-5	5	-	3	+	-6	0	-7	2	3	-3	14	
(15+15i)	450	+	+	+	-	2	2	6	6	-6	-6	2	2	-6	-6	2	
(21+3i)	450	-	1	+	+	-6	-1	3	-7	0	5	3	8	12	2	-5	
(22)	484	-	-3	-3	-5	-4	-4	6	6	0	0	-1	-1	0	0	-10	- (11)
(16+16i)	512	-	-2	2	2	-2	2	2	2	-6	6	10	-10	-6	-6	2	
(16+16i)	512	-	2	-2	2	2	-2	2	2	6	-6	-10	10	-6	-6	2	
(32)	1024	-	2	2	-2	2	2	-2	-2	-6	-6	10	10	-6	-6	2	
(32)	1024	+	-2	-2	-2	-2	-2	-2	-2	6	6	-10	-10	-6	-6	2	
(64)	4096	-	0	0	2	0	0	6	6	0	0	0	0	6	6	-14	
(64)	4096	+	0	0	-2	0	0	-6	-6	0	0	0	0	6	6	-14	

Table 5.1.3 Rational Newforms in $V^{-}(a)$

a	Na	1+i	2+i	2-i	3	3+2i	3-2i	1+4i	1-4i	5+2i	5-2i	1+6i	1-6i	5+4i	5-4i	7	Other W
(8+2i)	68	-	2	-2	2	-6	2	2	-	-6	-2	2	-2	2	-6	2	
(11+4i)	137	1	-2	0	-2	0	-2	-6	2	-6	-8	8	8	-2	-10	2	+ (11+4i)
(12)	144	-	-2	-2	-	-2	-2	2	2	6	6	6	6	-6	-6	-14	
(9+8i)	145	1	-	-2	-2	-6	-2	-6	2	6	-	-2	-2	-10	-6	-6	
(9+8i)	145	-2	+	-2	4	0	-2	6	2	-6	-	-2	10	8	-6	6	
(12+i)	145	0	2	-	4	-2	0	4	-2	-10	+	10	-4	-4	10	-2	
(12+4i)	160	+	-	-2	2	-2	-2	2	-6	-2	-2	-2	6	-6	10	2	
(12+6i)	180	-	2	+	-	6	-2	-6	2	-6	2	6	-2	-6	10	2	
(13+4i)	185	-2	0	-	2	-2	6	6	-2	-2	2	+	-8	0	-4	6	
(15+9i)	306	-	4	2	+	-4	0	-	2	-8	-6	10	-4	6	2	2	
(15+10i)	325	1	+	+	2	+	-6	-6	6	-6	2	-6	-2	2	-2	6	
(17+6i)	325	1	-2	-	0	0	-	-2	-2	-8	-4	-2	-6	2	8	-6	
(18+i)	325	0	+	0	2	4	+	6	0	-6	6	-2	2	0	-6	4	
(18+2i)	328	+	-2	-4	0	-6	2	0	-6	-4	-6	4	2	6	-	-2	
(19)	361	-1	2	2	1	1	1	-3	-3	-9	-9	-10	-10	6	6	-5	+ (19)
(19)	361	1	-2	-2	1	-5	-5	-3	-3	-3	-3	2	2	-6	-6	-5	+ (19)
(19+i)	362	-	-4	-1	2	-1	0	-2	5	3	-10	-10	-5	10	-7	-8	+ (9+10i)
(15+12i)	369	1	0	0	+	2	0	-2	-6	-6	-8	-4	0	+	2	10	
(17+9i)	370	-	+	-2	4	-2	4	4	-4	6	-6	6	+	6	10	-10	
(16+11i)	377	-2	3	4	-2	3	-	3	-4	-3	+	1	7	-7	-3	-12	
(15+13i)	394	-	0	-2	5	-5	-5	-3	-4	-7	-4	-3	8	11	-5	9	+ (1+14i)
(20)	400	-	+	+	-6	-2	-2	2	2	-2	-2	6	6	-6	-6	2	

Table 5.1.3 (Concluded)

a	Na	1+i	2+i	2−i	3	3+2i	3−2i	1+4i	1−4i	5+2i	5−2i	1+6i	1−6i	5+4i	5−4i	7	Other W
(20)	400	+	−	−	−2	2	2	−6	−6	6	6	2	2	6	6	−10	
(19+7i)	410	−	0	+	−4	−6	−6	2	4	−6	−8	2	−8	−	6	6	
(18+10i)	424	+	−1	−3	−5	4	−3	−4	−4	6	1	−6	6	−6	−3	5	+ (2+7i)
(19+8i)	425	−2	0	+	4	0	2	+	0	6	2	4	6	−8	4	4	
(19+9i)	442	+	−4	1	−1	3	+	−7	+	−6	−8	5	−2	5	−3	2	
(21+3i)	450	−	−1	−	−	−6	1	−3	−7	0	5	3	−8	−12	−2	5	
(16+14i)	452	−	2	4	−2	4	−4	−2	−6	4	2	2	4	6	−10	−10	+ (8+7i)
(17+13i)	458	−	1	0	4	6	6	1	1	0	−8	−8	−3	2	−11	8	+ (2+15i)
(18+12i)	468	−	0	−2	+	+	−6	4	−8	0	0	−10	2	6	12	−6	
(18+13i)	493	−1	1	0	−5	−2	6	−7	−	−	−3	−8	4	0	2	−6	
(20+10i)	500	−	+	+	2	2	−2	6	−6	6	6	2	−2	−6	−6	10	

Table 5.1.4 Elliptic Curves over Q($\sqrt{-1}$) with small conductor

The following region was searched: $a_1, a_3 \in \{0,1,i,1+i\}$; $a_2 \in \{x+iy : -1 \le x,y \le 1\}$; $a_4, a_6 \in \{x+iy : -3 \le x,y \le 3\}$. The table gives all curves found with conductor less than 500 in norm, except that curves isomorphic or conjugate to listed curves are omitted. Isogenies are indicated between listed curves: they are 2-isogenies unless otherwise indicated. In the column headed "CM & j", an entry "CM(n)" indicates that the curve has complex multiplication by an order in Q($\sqrt{-n}$); in this case, the j invariant is also given.

	#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies	
	1	1+i	i	i	0	0	11+2i	125	CM(1) 12^3	(3-4i)	25		
32A	2	0	0	0	-1	0	64	4096	CM(1) 12^3	(8)	64	•	
32D		1+i	i	1+i	2-i	-i	-8	64	CM(1) 66^3	(8)	64	•	
	3	1+i	0	1	0	0	-29+2i	845		(7+4i)	65	•	
		1+i	1+i	i	-1+i	-1	-1+18i	325		(7+4i)	65	•	
48A	4	0	1	0	1	0	-48	2304		(6+6i)	72	•	
48B		1+i	0	1+i	1-i	0	-36	1296		(6+6i)	72	•	
14A	5	i	0	i	0	0	-28	784		(7+7i)	98		
20A	6	0	1	0	-1	0	80	6400		(10)	100	•	
20B		1+i	-i	1+i	-1-i	0	100	10000		(10)	100	•	
	7	1	-1+i	1+i	-1-i	0	144+80i	27136		(9+5i)	106		
	8	0	-1	1	0	0	-11	121		(11)	121		
	9	i	1-i	i	-i	0	-16+2i	260		(9+7i)	130		
	10	0	-1-i	0	1	-1-i	192-256i	102400		(12+4i)	160	•	← f= (8+16i)
		0	-1+i	0	1+2i	-3-i	-64-128i	204800		(12+4i)	160	•	
		1-i	1+i	1-i	-5+4i	2-3i	16-8i	320		(12+4i)	160	•	

Table 5.1.4 (Continued)

#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies
11	1+i	1+i	0	i	0	20+16i	656		(10+8i)	164	
12	1	-1+i	-i	-4-3i	-4i	20-52i	3104		(13+5i)	194	
13	-i	i	-i	-i	1	-104-40i	12416		(13+5i)	194	
14	1+i	i	0	1	0	-100	10000		(10+10i)	200	
	0	0	0	-2	-1	80	6400		(10+10i)	200	
	1+i	i	1+i	-4-i	2i	10000	100000000		(10+10i)	200	
15	1	1	1	0	0	-15	225		(15)	225	
	1	1	1	-5	2	225	50625		(15)	225	
16	0	1-i	i	-i	0	13+8i	233		(13+8i)	233	
17	0	0	0	1	0	-64	4096	CM(1) 12^3	(16)	256	
18	1	i	-i	i	0	1-16i	257		(16+i)	257	
19	1-i	0	-i	-1+i	-i	731-542i	828125		(16+3i)	265	
	1-i	-1-i	1	-4+2i	3+i	439-398i	351125		(16+3i)	265	
20	1-i	1+i	1-i	-1+i	-1	-60-32i	4624		(16+4i)	272	
	0	1+i	0	2i	-1	16-64i	4352		(16+4i)	272	
21	1	1+i	1+i	i	0	14+9i	277		(14+9i)	277	
22	i	1	i	0	0	17	289		(17)	289	
23	1	1	-i	1	1	-64-72i	9280		(17+i)	290	
24	1	1+i	i	0	0	-22+12i	628		(17+5i)	314	
25	0	0	0	0	1	-432	186624		(18)	324	
	1+i	i	0	3	-i	-108	11664		(18)	324	
26	1	0	1	0	0	-26	676		(13+13i)	338	

Table 5.1.4 (Concluded)

#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies
27	i	1	i	-2	-3	-1664	2768896		(13+13i)	338	
28	0	i	0	-1+2i	-1-i	-304+128i	108800		(14+12i)	340	
	1-i	-1-i	1-i	1-i	-4+i	124-88i	23120		(14+12i)	340	
29	0	1	1	1	0	-19	361		(19)	361	
30	1	-1+i	1	i	0	-38-2i	1448		(19+i)	362	
31	0	-i	i	i	1	-11+16i	377		(16+11i)	377	
32	1+i	-i	1+i	-i	0	112	12544		(14+14i)	392	
33	1+i	i	1+i	4-i	-2i	-784	614656		(14+14i)	392	
	1+i	i	0	-1	0	28	784		(14+14i)	392	
34	1	1-i	1	-i	0	16-62i	4100		(17+11i)	410	
35	1	-i	-i	1+i	-i	16-512i	262400		(19+7i)	410	
36	1	0	0	-4	-1	3969	15752961		(21)	441	
	1	0	0	1	0	-63	3969		(21)	441	
37	i	-1	0	2-i	i	19+9i	442		(19+9i)	442	
38	1	0	1	1	2	-2160	4665600		(15+15i)	450	
39	-i	-i+i	1-i	0	0	12-84i	7200		(21+3i)	450	
40	0	-i	1+i	-1	0	44	1936		(22)	484	

Missing conductors: (17+11i), (~~19+8i~~).

Table 5.1.4a Elliptic Curves over Q(i) with conductor $(1+i)^e$ for $e \le 12$

This list is taken from [19](note that Stroeker's curves II3, II4, III3, and III4 have conductor $(1+i)^{12}$, and not $(1+i)^{10}$ as he asserts). Only one curve from each isogeny class is listed. There are also 8 isogeny classes of curves with conductor $(1+i)^{13}$ and 12 with conductor $(1+i)^{14}$. The symbols used to identify the curves are the ones used by Stroeker.

Symbol	a_1	a_2	a_3	a_4	a_6	Δ	CM	j	Conductor
I1	0	0	0	-1	0	2^6	CM(1)	12^3	$(1+i)^6$
I2	0	0	0	1	0	-2^6	CM(1)	12^3	$(1+i)^8$
IX1	0	1+i	0	i	0	$i2^5$		2^7	$(1+i)^9$
IX2	0	1-i	0	-i	0	$-i2^5$		2^7	$(1+i)^9$
I3	0	0	0	2i	0	$i2^9$	CM(1)	12^3	$(1+i)^{10}$
I4	0	0	0	-2i	0	$-i2^9$	CM(1)	12^3	$(1+i)^{10}$
V1	0	1	0	-2	-2	2^7		$2^5 7^3$	$(1+i)^{10}$
V2	0	i	0	2	2i	-2^7		$2^5 7^3$	$(1+i)^{10}$
II1	0	0	0	i	0	$i2^6$	CM(1)	12^3	$(1+i)^{12}$
II2	0	0	0	-i	0	$-i2^6$	CM(1)	12^3	$(1+i)^{12}$
II3	0	0	0	-2	0	2^9	CM(1)	12^3	$(1+i)^{12}$
II4	0	0	0	2	0	-2^9	CM(1)	12^3	$(1+i)^{12}$
III1	0	1+i	0	-i	1-i	$-i2^6$	CM(2)	20^3	$(1+i)^{12}$
III3	0	1	0	-3	1	2^9	CM(2)	20^3	$(1+i)^{12}$

Table 5.1.5 Elliptic Curves over $Q(\sqrt{-1})$: Zeta functions

For each isogeny class of curves in the previous table (5.1.4), the first fifteen coefficients a_π (for π prime) of the zeta function are given: a_π is the Trace of Frobenius at π. (Isogenous curves have the same zeta function.)

	#	f	Nf	1+i	2+i	2-i	3	3+2i	3-2i	1+4i	1-4i	5+2i	5-2i	1+6i	1-6i	5+4i	5-4i	7
	1	(3-4i)	25	-2	0	-4	0	4	-6	-2	8	-10	10	-12	-2	-8	-8	0
	2	(8)	64	0	-2	-2	-6	6	6	2	2	-10	-10	-2	-2	10	10	-14
	3	(7+4i)	65	0	-1	0	-2	-4	1	6	0	-6	6	2	2	0	6	-4
48A	4	(6+6i)	72	0	-2	-2	1	-2	-2	2	2	6	6	6	6	-6	-6	-14
14C	5	(7+7i)	98	-1	0	0	-2	-4	-4	6	6	-6	-6	2	2	6	6	1
20A	6	(10)	100	0	-1	-1	-2	2	2	-6	-6	6	6	2	2	6	6	-10
	7	(9+5i)	106	1	-3	-3	1	-4	5	0	0	-6	3	2	2	6	-3	5
11A	8	(11)	121	-2	1	1	-5	4	4	-2	-2	0	0	3	3	-8	-8	-10
	9	(9+7i)	130	-1	0	-1	4	-4	1	-6	0	-6	0	2	2	-6	6	14
	10	(12+4i)	160	0	1	-2	2	-2	-2	2	-6	-2	-2	-2	6	-6	10	2
	11	(10+8i)	164	0	0	0	-2	2	-4	6	-6	6	0	8	-4	-1	-6	2
	12	(13+5i)	194	-1	3	-3	4	-4	-4	3	0	6	-3	-7	-7	0	-9	-4
	13	(13+5i)	194	1	-1	-1	-4	0	0	-3	4	2	9	3	-11	0	7	8
40A	14	(10+10i)	200	0	1	1	-6	-2	-2	2	2	-2	-2	6	6	-6	-6	2
15A	15	(15)	225	-1	1	1	1	-2	-2	2	2	-2	-2	-10	-10	10	10	-14
	16	(13+8i)	233	-2	-3	-2	-4	3	-3	-4	-3	6	-7	4	4	12	-5	1
	17	(16)	256	0	2	2	-6	-6	-6	2	2	10	10	2	2	10	10	-14
	18	(16+i)	257	-1	-2	2	2	6	-2	2	2	-6	-6	-10	-6	10	2	2
	19	(16+3i)	265	0	0	1	4	2	2	-6	-6	6	-6	2	-4	0	-6	2
	20	(16+4i)	272	0	2	-2	2	-6	2	2	1	-6	-2	2	-2	2	-6	2
	21	(14+9i)	277	-1	-4	-1	-1	-4	-6	-1	2	1	5	-6	-1	6	6	-1
17C	22	(17)	289	-1	-2	-2	-6	-2	-2	1	1	6	6	-2	-2	-6	-6	2

Table 5.1.5 (Concluded)

	#	f	Nf	1+i	2+i	2-i	3	3+2i	3-2i	1+4i	1-4i	5+2i	5-2i	1+6i	1-6i	5+4i	5-4i	7
	23	(17+i)	290	1	0	-1	-2	-4	2	0	0	-1	0	-10	8	-6	6	-10
	24	(17+5i)	314	-1	-4	-3	-1	-3	-1	-7	5	2	-6	-2	-2	-5	4	4
36A	25	(18)	324	0	0	0	0	2	2	0	0	0	0	-10	-10	0	0	2
26A	26	(13+13i)	338	-1	-3	-3	-5	1	1	-3	-3	6	6	-7	-7	0	0	-13
26D	27	(13+13i)	338	1	-1	-1	3	-1	-1	-3	-3	2	2	3	3	0	0	-13
	28	(14+12i)	340	0	0	-1	4	2	-4	6	-1	-6	-6	-10	2	0	-6	-4
9A	29	(19)	361	0	3	3	-2	-4	-4	-3	-3	6	6	2	2	-6	-6	-13
	30	(19+i)	362	-1	0	3	-2	5	-4	-6	3	-9	6	2	-7	6	9	-4
	31	(16+11i)	377	0	3	0	-2	5	1	-3	0	3	-1	-7	-7	-9	-3	-4
56A	32	(14+14i)	392	0	-4	-4	-2	0	0	-2	-2	2	2	-6	-6	-2	-2	1
56C	33	(14+14i)	392	0	2	2	-6	2	2	-6	-6	6	6	-2	-2	2	2	1
	34	(17+11i)	410	-1	-4	-1	-4	0	-6	0	2	-6	-2	-2	8	6	-1	-2
	35	(19+7i)	410	1	-2	1	2	-2	-2	2	-6	-2	6	-10	-2	1	10	2
21A	36	(21)	441	-1	-2	-2	1	-2	-2	-6	-6	-2	-2	6	6	2	2	1
	37	(19+9i)	442	-1	0	3	-5	5	1	3	-1	-6	0	-7	2	3	-3	14
30A	38	(15+15i)	450	-1	-1	-1	1	2	2	6	6	-6	-6	2	2	-6	-6	2
	39	(21+3i)	450	1	1	0	-1	-6	-1	3	-7	0	5	3	8	12	2	-5
44A	40	(22)	484	0	-3	-3	-5	-4	-4	6	6	0	0	-1	-1	0	0	-10

Table 5.1.5a Curves with Conductor $(1+i)^e$ for $e \le 12$: Zeta Functions

For each of the curves in Table 5.1.4a we give the same information as in Table 5.1.5.

#	f	Nf	1+i	2+i	2-i	3	3+2i	3-2i	1+4i	1-4i	5+2i	5-2i	1+6i	1-6i	5+4i	5-4i	7
I1	$(1+i)^6$	2^6	0	-2	-2	-6	6	6	2	2	-10	-10	-2	-2	10	10	-14
I2	$(1+i)^8$	2^8	0	2	2	-6	-6	-6	2	2	10	10	2	2	10	10	-14
IX1	$(1+i)^9$	2^9	0	-2	2	2	-2	2	2	2	-6	6	10	-10	-6	-6	2
IX2	$(1+i)^9$	2^9	0	2	-2	2	2	-2	2	2	6	-6	-10	10	-6	-6	2
I3	$(1+i)^{10}$	2^{10}	0	-2	2	6	6	-6	-2	-2	10	-10	2	-2	10	10	-14
I4	$(1+i)^{10}$	2^{10}	0	2	-2	6	-6	6	-2	-2	-10	10	-2	2	10	10	-14
V1	$(1+i)^{10}$	2^{10}	0	2	2	-2	2	2	-2	-2	-6	-6	10	10	-6	-6	2
V2	$(1+i)^{10}$	2^{10}	0	-2	-2	-2	-2	-2	-2	-2	6	6	-10	-10	-6	-6	2
II1	$(1+i)^{12}$	2^{12}	0	4	-4	6	4	-4	2	2	-4	4	12	-12	-10	-10	-14
II2	$(1+i)^{12}$	2^{12}	0	-4	4	6	-4	4	2	2	4	-4	-12	12	-10	-10	-14
II3	$(1+i)^{12}$	2^{12}	0	-4	-4	-6	-4	-4	-2	-2	-4	-4	12	12	-10	-10	-14
II4	$(1+i)^{12}$	2^{12}	0	4	4	-6	4	4	-2	-2	4	4	-12	-12	-10	-10	-14
III1	$(1+i)^{12}$	2^{12}	0	0	0	2	0	0	6	6	0	0	0	0	6	6	-14
III3	$(1+i)^{12}$	2^{12}	0	0	0	-2	0	0	-6	-6	0	0	0	0	6	6	-14

§5.2 The results of Computations for $Q(\sqrt{-2})$

Table 5.2.1 Ideals $\mathfrak{a}$ of $Z[\theta]$ with $N\mathfrak{a} \leq 300$ and $\dim V(\mathfrak{a}) > 0$

Table 5.2.2 Rational Newforms in $V^{+}(\mathfrak{a})$

Table 5.2.3 Rational Newforms in $V^{-}(\mathfrak{a})$

Table 5.2.4 Elliptic Curves with small conductor

Table 5.2.5 Elliptic Curves with small conductor: Zeta Functions

Table 5.2.1 Ideals a of $Z[\theta]$ with Na $\leq$ 300 and dim V(a) > 0

Only one ideal of each conjugate pair is listed. The splitting field is Q unless otherwise stated. Numbers in parentheses give the dimension of the appropriate subspace of newforms, when this is less than that of the whole space.

a	Na	dim V(a)	dim V^+(a)	dim V^-(a)	Splitting Field
(4θ)	32	1	1	0	
(6)	36	1	0	1	
$(3 + 4\theta)$	41	1	0	1	
(7)	49	1	0	1	
$(7 + \theta)$	51	1	1	0	
$(2 + 5\theta)$	54	2	1	1	
(8)	64	2 (0)	2 (0)	0	
$(8 + \theta)$	66	1	0	1	
(6θ)	72	3 (1)	1	2 (0)	
$(5 + 5\theta)$	75	1	0	1	
$(8 + 3\theta)$	82	2 (0)	0	2 (0)	
$(8 + 4\theta)$	96	2 (0)	2 (0)	0	
(7θ)	98	3 (1)	1	2 (0)	
$(9 + 3\theta)$	99	1	1	0	
(10)	100	3	1	2	$Q(\sqrt{3})$
$(2 + 7\theta)$	102	2 (0)	2 (0)	0	
$(10 + 2\theta)$	108	4 (0)	2 (0)	2 (0)	
$(6 + 6\theta)$	108	3 (1)	1	2 (0)	
$(4 + 7\theta)$	114	2	1	1	
(11)	121	2	1	1	
$(11 + \theta)$	123	3 (1)	0	3 (1)	
$(5 + 7\theta)$	123	2 (0)	0	2 (0)	
(8θ)	128	5 (2)	3 (0)	2	
$(11 + 2\theta)$	129	2	0	2	
$(10 + 4\theta)$	132	1	0	1	
$(2 + 8\theta)$	132	2 (0)	0	2 (0)	
$(6 + 7\theta)$	134	1	0	1	
$(8 + 6\theta)$	136	1	0	1	
(12)	144	7 (2)	3 (1)	4 (1)	
$(12 + \theta)$	146	1	0	1	
$(7 + 7\theta)$	147	2 (0)	0	2 (0)	

Table 5.2.1 (Continued)

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
$(10 + 5\theta)$	150	2 (0)	0	2 (0)	
$(9 + 6\theta)$	153	2 (0)	2 (0)	0	
$(5 + 8\theta)$	153	3 (1)	2 (0)	1	
$(12 + 3\theta)$	162	4 (0)	2 (0)	2 (0)	
$(8 + 7\theta)$	162	4 (0)	2 (0)	2 (0)	
$(6 + 8\theta)$	164	4 (1)	1	3 (0)	
$(2 + 9\theta)$	166	1	0	1	
(13)	169	2	0	2	$Q(\sqrt{2})$
$(3 + 9\theta)$	171	1	0	1	
$(4 + 9\theta)$	178	1	1	0	
$(8 + 8\theta)$	192	4 (0)	4 (0)	0	
$(12 + 5\theta)$	194	2	1	1	
(14)	196	7 (2)	2 (0)	5 (2)	$Q(\sqrt{3})$
$(10 + 7\theta)$	198	3 (1)	1	2 (0)	
$(6 + 9\theta)$	198	5 (1)	2 (0)	3 (1)	
(10θ)	200	7 (1)	3 (1)	4 (0)	
$(14 + 2\theta)$	204	3 (0)	3 (0)	0	
$(2 + 10\theta)$	204	2	0	2	
$(12 + 6\theta)$	216	9 (1)	4 (0)	5 (1)	
$(4 + 10\theta)$	216	6 (0)	3 (0)	3 (0)	
$(11 + 7\theta)$	219	1	1	0	
(15)	225	7 (3)	1	6 (2)	
$(5 + 10\theta)$	225	3 (1)	1	2 (0)	
$(14 + 4\theta)$	228	4 (0)	2 (0)	2 (0)	
$(12 + 7\theta)$	242	2	1	1	
(11θ)	242	4 (0)	2 (0)	2 (0)	
$(14 + 5\theta)$	246	4 (0)	0	4 (0)	
$(2 + 11\theta)$	246	9 (3)	1	8 (2)	
$(7 + 10\theta)$	249	1	1	0	
(16)	256	11 (3)	6 (2)	5 (1)	
$(16 + \theta)$	258	1	1	0	
$(4 + 11\theta)$	258	4 (0)	0	4 (0)	
$(16 + 2\theta)$	264	5 (2)	1	4 (1)	
$(8 + 10\theta)$	264	2 (0)	0	2 (0)	
$(13 + 7\theta)$	267	2	2	0	$Q(\sqrt{5})$

Table 5.2.1 (Concluded)

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
$(5 + 11\theta)$	267	1	1	0	
$(14 + 6\theta)$	268	2 (0)	0	2 (0)	
$(12 + 8\theta)$	272	3 (1)	0	3 (1)	
$(6 + 11\theta)$	278	1	0	1	
$(16 + 4\theta)$	288	4 (1)	3 (0)	1	
(12θ)	288	19 (4)	13 (4)	6 (0)	
(17)	289	5	1	4	$Q(\sqrt{3})$
$(2 + 12\theta)$	292	2 (0)	0	2 (0)	
$(14 + 7\theta)$	294	9 (3)	2 (0)	7 (3)	
$(15 + 6\theta)$	297	5 (3)	3 (1)	2	
$(13 + 8\theta)$	297	4	2	2	
$(3 + 12\theta)$	297	5 (3)	3 (1)	2	
$(10 + 10\theta)$	300	9 (0)	2 (0)	7 (0)	

Table 5.2.2 $\mathbb{Z}[\theta]$: Rational Newforms in $V^+(a)$

a	Na	θ	1+θ	1−θ	3+θ	3−θ	3+2θ	3−2θ	1+3θ	1−3θ	5	3+4θ	3−4θ	5+3θ	5−3θ	7	Other W
(4θ)	32	−	0	0	0	0	2	2	0	0	−6	10	10	0	0	−14	
(7+θ)	51	−2	+	−2	−2	−4	−6	+	−4	0	−2	−2	−2	10	−2	−2	
(2+5θ)	54	+	−	1	3	0	−6	−3	2	2	−1	0	−3	8	−10	−4	
(6θ)	72	−	+	+	4	4	2	2	−4	−4	−6	−6	−6	4	4	−14	
(7θ)	98	+	−2	−2	0	0	6	6	2	2	−10	6	6	8	8	−	
(9+3θ)	99	−1	+	−	+	4	−6	2	4	4	2	−6	−6	12	4	−6	
(10)	100	−	−2	−2	0	0	−6	−6	−4	−4	−	6	6	−10	−10	−10	
(6+6θ)	108	−	−	−	0	0	−6	6	−4	−4	2	−6	6	−4	−4	2	
(4+7θ)	114	−	+	0	4	4	−2	−6	0	+	−2	−10	6	−4	0	2	
(11)	121	−2	−1	−1	−	−	−2	−2	0	0	−9	−8	−8	−6	−6	−10	
(12)	144	−	−	−	−4	−4	2	2	4	4	−6	−6	−6	−4	−4	−14	
(6+8θ)	164	−	0	−2	−4	−6	2	−2	4	−2	−2	−	−6	−2	−2	10	
(4+9θ)	178	−	1	1	0	−3	−6	3	−4	5	8	0	−9	8	8	2	+ (9−2θ)
(12+5θ)	194	+	0	−1	3	−6	−5	−6	−4	6	−3	−6	10	4	4	2	+ (5−6θ)
(10+7θ)	198	+	−	−2	−	−6	0	0	−4	−4	−4	6	−6	−4	2	−10	
(10θ)	200	+	0	0	4	4	2	2	4	4	−	−6	−6	−8	−8	2	
(11+7θ)	219	0	0	+	−2	0	−2	−2	−2	−6	−2	6	−6	4	−10	−2	+ (1−6θ)
(15)	225	−1	+	+	−4	−4	2	2	4	4	−	10	10	4	4	−14	
(5+10θ)	225	2	0	−	2	6	−2	2	−6	2	+	−6	6	−4	−8	−6	
(12+7θ)	242	+	−1	2	3	−	0	6	−2	5	4	0	6	4	−4	−5	
(2+11θ)	246	+	−	−3	−3	−4	1	−7	−5	1	4	−12	−	−2	1	−11	

Table 5.2.2 (Concluded)

a	Na	θ	$1+\theta$	$1-\theta$	$3+\theta$	$3-\theta$	$3+2\theta$	$3-2\theta$	$1+3\theta$	$1-3\theta$	5	$3+4\theta$	$3-4\theta$	$5+3\theta$	$5-3\theta$	7	Other W
$(7+10\theta)$	249	-1	+	1	-2	-5	-1	-7	2	4	-6	9	-11	-7	11	-4	+ $(9+\theta)$
(16)	256	–	2	2	-2	-2	-2	-2	2	2	-6	-6	-6	6	6	2	
(16)	256	+	-2	-2	2	2	-2	-2	-2	-2	-6	-6	-6	-6	-6	2	
$(16+\theta)$	258	–	+	2	4	-2	-6	0	-6	4	-2	0	6	+	-2	-6	
$(16+2\theta)$	264	+	2	+	2	–	-4	0	8	4	-8	2	-2	10	-12	2	
$(5+11\theta)$	267	0	–	-2	0	0	-6	6	2	8	2	-6	6	2	8	-10	+ $(9+2\theta)$
(12θ)	288	–	–	–	4	4	-6	-6	-4	-4	-6	2	2	4	4	2	
(12θ)	288	+	+	+	-4	-4	-6	-6	4	4	-6	2	2	-4	-4	2	
(12θ)	288	+	+	–	4	-4	2	2	4	-4	10	-6	-6	4	-4	2	
(12θ)	288	+	–	+	-4	4	2	2	-4	4	10	-6	-6	-4	4	2	
(17)	289	-1	0	0	0	0	–	–	-4	-4	-6	-6	-6	4	4	2	
$(15+6\theta)$	297	0	+	–	0	+	6	6	-4	2	2	0	12	-4	2	-10	
$(13+8\theta)$	297	0	-2	–	+	3	3	-3	-7	2	8	6	12	-1	8	-4	
$(13+8\theta)$	297	2	2	–	+	-5	3	5	1	-2	-4	6	0	-9	4	12	
$(3+12\theta)$	297	2	+	–	+	4	0	-4	-2	-8	2	6	6	6	-8	6	

Table 5.2.3 $Z[\theta]$: Rational Newforms in $V^{-}(a)$

a	Na	θ	$1+\theta$	$1-\theta$	$3+\theta$	$3-\theta$	$3+2\theta$	$3-2\theta$	$1+3\theta$	$1-3\theta$	5	$3+4\theta$	$3-4\theta$	$5+3\theta$	$5-3\theta$	7	Other W
(6)	36	−	+	+	0	0	−6	−6	−4	−4	−2	6	6	4	4	2	
$(3+4\theta)$	41	2	−2	0	−2	0	−6	2	−6	−4	−2	+	−10	8	4	6	
(7)	49	0	−2	−2	−2	−2	−2	−2	6	6	−6	−10	−10	2	2	+	
$(2+5\theta)$	54	−	+	−1	−3	0	−6	3	2	2	1	0	−3	−8	10	−4	
$(8+\theta)$	66	+	2	+	−6	+	0	0	−4	−4	4	−6	−6	−2	4	−10	
$(5+5\theta)$	75	−2	+	0	−6	2	2	2	2	−6	−	−6	−6	8	4	−6	
$(4+7\theta)$	114	+	+	2	0	0	0	−6	2	−	−8	0	−12	−8	10	14	
(11)	121	0	−1	−1	+	+	−6	−6	−4	−4	7	0	0	−2	−2	2	
$(11+\theta)$	123	−2	−2	−	6	−4	2	2	2	−8	−2	+	6	−4	0	−10	
(8θ)	128	−	2	−2	−2	2	−2	−2	−2	2	−6	6	6	−6	6	−2	
(8θ)	128	+	−2	2	2	−2	−2	−2	2	−2	−6	6	6	6	−6	−2	
$(11+2\theta)$	129	0	+	−1	3	−6	−3	0	−7	2	1	−6	3	−8	+	−4	
$(11+2\theta)$	129	2	+	3	−1	−2	−3	4	5	−6	−7	−2	3	12	+	8	
$(10+4\theta)$	132	−	−	0	−4	−	2	−2	−4	4	−6	−10	−2	8	4	6	
$(6+7\theta)$	134	−	−1	−1	3	0	0	6	2	−7	−8	3	0	−8	10	5	− $(7-3\theta)$
$(8+6\theta)$	136	+	2	0	−4	−2	−6	−	−6	−2	2	2	6	12	−8	−10	
(12)	144	+	−	−	0	0	−6	−6	4	4	−2	6	6	−4	−4	2	
$(12+\theta)$	146	−	−2	0	4	0	0	−2	−4	8	4	−4	−12	6	−4	−2	− $(1-6\theta)$
$(5+8\theta)$	153	2	−	2	2	−4	−6	−	−4	0	2	2	−2	−10	2	−2	
$(2+9\theta)$	166	−	1	0	−1	−2	5	−1	−8	−6	−2	−9	7	−7	1	−8	− $(9-\theta)$
$(3+9\theta)$	171	−1	+	−	−4	0	6	−6	+	−8	2	−2	6	−8	−4	10	
$(12+5\theta)$	194	+	2	−1	−3	0	3	0	2	2	1	−6	0	−8	−8	−4	− $(5-6\theta)$

Table 5.2.3 (Concluded)

a	Na	θ	1+θ	1−θ	3+θ	3−θ	3+2θ	3−2θ	1+3θ	1−3θ	5	3+4θ	3−4θ	5+3θ	5−3θ	7	Other W
(6+9θ)	198	−	+	+	0	−	6	6	8	−4	−2	−6	6	−8	4	−10	
(12+6θ)	216	−	−	+	4	−4	−2	2	−4	−4	6	−6	6	−4	−4	−14	
(15)	225	0	+	+	0	0	0	0	2	2	+	−6	−6	−8	−8	14	
(15)	225	0	−	−	−4	−4	0	0	−6	−6	+	2	2	0	0	−2	
(12+7θ)	242	+	−1	2	−3	+	0	−6	2	5	4	0	−6	4	4	5	
(2+11θ)	246	+	−	3	1	−4	−3	−3	−3	−3	8	−4	+	10	1	−5	
(2+11θ)	246	−	+	−1	3	0	−3	3	−1	−7	4	12	−	−2	−5	5	
(16)	256	+	0	0	0	0	2	2	0	0	−6	−10	−10	0	0	14	
(16+2θ)	258	+	0	+	4	−	−6	2	0	−4	2	10	−10	−4	−4	6	
(12+8θ)	272	−	0	−2	2	4	−	−6	2	6	2	6	2	8	−12	−10	
(6+11θ)	278	−	−1	−2	−2	6	3	−2	4	−1	8	6	7	−6	1	−4	− (11−3θ)
(17)	289	0	0	0	−4	−4	−	−	2	2	6	−10	−10	−2	−2	−2	
(17)	289	0	−2	−2	2	2	+	+	−6	−6	2	6	6	2	2	6	
(14+7θ)	294	−	−1	+	3	0	0	3	2	2	1	6	6	1	1	−	
(14+7θ)	294	−	1	−	−5	4	4	1	6	−6	−3	−10	2	−7	5	+	
(14+7θ)	294	+	−2	−	4	4	−8	4	0	−6	−6	8	−4	8	−4	+	
(15+6θ)	297	0	+	+	0	+	6	−6	−4	2	−2	0	12	4	−2	−10	
(15+6θ)	297	1	−	−	−4	+	2	6	4	4	−2	6	−6	−4	−12	−6	
(13+8θ)	297	0	2	−	+	−3	−3	−3	−7	2	−8	6	−12	1	−8	−4	
(13+8θ)	297	2	−2	−	+	5	−3	5	1	−2	4	6	0	9	−4	12	
(3+12θ)	297	1	−	+	−	4	−6	−2	4	4	−2	6	−6	−12	−4	−6	
(3+12θ)	297	−2	+	+	−	4	0	4	−2	−8	−2	−6	6	−6	8	6	

Table 5.2.4 Elliptic Curves over $\mathbb{Q}(\sqrt{-2})$ with small conductor

The following region was searched: $a_1,a_3 \in \{0,1,\theta,1+\theta\}$; $a_2 \in \{x+y\theta:-1\le x,y\le 1\}$; $a_4,a_6 \in \{x+y\theta:-3\le x,y\le 3\}$.

	#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies
	1	θ	$1-\theta$	1	-1	0	$-23+10\theta$	729	CM(2) 20^3	$(1-2\theta)$	9	
64A	2	0	0	0	1	0	-64	4096	CM(1) 12^3	(4θ)	32	
		0	0	0	-1	0	64	4096	CM(1) 12^3	(4θ)	32	
		θ	-1	θ	-1	0	8	64	CM(1) 66^3	(4θ)	32	
		θ	-1	0	-2	3	8	64	CM(1) 66^3	(4θ)	32	
	3	$-\theta$	$-1+\theta$	1	0	0	$-47-14\theta$	2601		$(7+\theta)$	51	
		$-\theta$	$-1+\theta$	$1-\theta$	$2-\theta$	θ	$37-2\theta$	1377		$(7+\theta)$	51	
	4	$1-\theta$	θ	1	-1	0	$-22-\theta$	486		$(2+5\theta)$	54	
		1	$1-\theta$	0	$-1-\theta$	-1	$-4-10\theta$	216		$(2+5\theta)$	54	3
	5	θ	1	θ	0	0	36	1296		(6θ)	72	
24A		0	-1	0	1	0	-48	2304		(6θ)	72	
14A	6	1	0	1	-1	0	-28	784		(7θ)	98	
	7	$1+\theta$	$1-\theta$	$1+\theta$	-2θ	$-\theta$	$9+36\theta$	2673		$(9+3\theta)$	99	
20A	8	0	1	0	-1	0	80	6400		(10)	100	
		θ	0	θ	2	0	-100	10000		(10)	100	
	9	1	$-\theta$	θ	$-\theta$	0	$4+64\theta$	8208		$(4+7\theta)$	114	
11A	10	0	-1	1	0	0	-11	121		(11)	121	
	11	θ	1	1	$3-3\theta$	$-3-\theta$	$1169+450\theta$	1771561	CM(2) 20^3	$(7-6\theta)$	121	
	12	θ	0	θ	0	1	36	1296		(12)	144	
48A		0	1	0	1	0	-48	2304		(12)	144	

Table 5.2.4 (Continued)

	#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies
	13	$-\theta$	$1-\theta$	$-\theta$	2	$1-\theta$	$92-96\theta$	26896		$(6+8\theta)$	164	
		0	$-1-\theta$	0	$1+2\theta$	$2-\theta$	$-48-64\theta$	10496		$(6+8\theta)$	164	
	14	$1-\theta$	$-1-\theta$	$1-\theta$	$\cancel{-1+\theta}$ $-1+2\theta$	$1+\theta$	$-8-18\theta$	712		$(4+9\theta)$	178	
	15	1	θ	1	-1	0	$-12-5\theta$	194		$(12+5\theta)$	194	
	16	1	$1-\theta$	$-\theta$	$2+\theta$	$2-2\theta$	$3448-992\theta$	13856832		$(10+7\theta)$	198	3
		$1-\theta$	θ	$1-\theta$	$1+3\theta$	$2+2\theta$	$526-2384\theta$	11643588		$(10+7\theta)$	198	
40A	17	0	0	0	-2	1	80	6400		(10θ)	200	
		θ	-1	θ	0	2	100	10000		(10θ)	200	
	18	θ	θ	$1+\theta$	θ	0	$41+46\theta$	5913		$(11+7\theta)$	219	
15A	19	1	1	1	0	0	-15	225		(15)	225	
	20	$1-\theta$	0	$1-\theta$	$-\theta$	θ	$-598-1198\theta$	3228012		$(2+11\theta)$	246	
	21	1	$-1+\theta$	1	$-1+\theta$	$-2-\theta$	$-55-43\theta$	6723		$(7+10\theta)$	249	
	22	0	θ	0	1	θ	-64	4096	CM(2) 20^3	(16)	256	
	23	0	$-\theta$	0	1	$-\theta$	-64	4096	CM(2) 20^3	(16)	256	
128H	24	0	-1	0	-2	2	128	16384		(16)	256	
128A		0	-1	0	1	-1	-256	65536		(16)	256	
128E	25	0	1	0	-2	-2	128	16384		(16)	256	
128C		0	1	0	1	1	-256	65536		(16)	256	
	26	$1-\theta$	$-\theta$	$-\theta$	1	$-1-\theta$	$142-104\theta$	41796		$(16+\theta)$	258	
	27	$-\theta$	1	$-\theta$	$-1-2\theta$	$-1-2\theta$	$-800-1024\theta$	2737152		$(16+2\theta)$	264	
	28	$-\theta$	0	$1-\theta$	$-\theta$	$-1-2\theta$	$4801-3482\theta$	47298249		$(5+11\theta)$	267	

Table 5.2.4 (Concluded)

#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies
96E 29	0	1	0	-2	0	576	331776		(12θ)	288	
	θ	0	0	2	-1	-648	419904		(12θ)	288	
96A 30	0	-1	0	-2	0	576	331776		(12θ)	288	
31	0	θ	0	1	3θ	5184	26873856		(12θ)	288	
32	0	$-\theta$	0	1	-3θ	5184	26873856		(12θ)	288	
17A 33	1	-1	1	-1	0	17	289		(17)	289	
34	θ	$1-\theta$	$1+\theta$	$-1-\theta$	1	$-459+54\theta$	216513		$(15+6\theta)$	297	
	$-\theta$	$1-\theta$	$1-\theta$	$1+2\theta$	0	$189-162\theta$	88209		$(15+6\theta)$	297	
35	0	$-1+\theta$	1	$1-\theta$	θ	$13+8\theta$	297		$(13+8\theta)$	297	
36	0	$1-\theta$	1	$2-2\theta$	1	$1117-592\theta$	1948617		$(13+8\theta)$	297	

Missing conductors: $(\cancel{6+6\theta})$, $(\cancel{5+10\theta})$, $(12+7\theta)$, $(\cancel{3+12\theta})$.

Table 5.2.5 Elliptic Curves over $Q(\sqrt{-2})$: Zeta Functions

For each isogeny class of curves in Table 5.2.4, we give the Trace of Frobenius at the first fifteen primes.

#	f	Nf	θ	1+θ	1−θ	3+θ	3−θ	3+2θ	3−2θ	1+3θ	1−3θ	5	3+4θ	3−4θ	5+3θ	5−3θ	7
1	(1−2θ)	9	0	0	−2	−6	6	6	−6	2	2	−10	−6	6	−10	−10	14
2	(4θ)	32	0	0	0	0	0	2	2	0	0	−6	10	10	0	0	−14
3	(7+θ)	51	−2	−1	−2	−2	−4	−6	−1	−4	0	−2	−2	−2	10	−2	−2
4	(2+5θ)	54	−1	0	1	3	0	−6	−3	2	2	−1	0	−3	8	−10	−4
5	(6θ)	72	0	−1	−1	4	4	2	2	−4	−4	−6	−6	−6	4	4	−14
6	(7θ)	98	−1	−2	−2	0	0	6	6	2	2	−10	6	6	8	8	1
7	(9+3θ)	99	−1	−1	1	−1	4	−6	2	4	4	2	−6	−6	12	4	−6
8	(10)	100	0	−2	−2	0	0	−6	−6	−4	−4	1	6	6	−10	−10	−10
9	(4+7θ)	114	1	−1	0	4	4	−2	−6	0	−1	−2	−10	6	−4	0	2
10	(11)	121	−2	−1	−1	1	1	−2	−2	0	0	−9	−8	−8	−6	−6	−10
11	(7−6θ)	121	0	2	2	−6	0	6	−6	−2	2	10	6	−6	10	−10	−14
12	(12)	144	0	1	1	−4	−4	2	2	4	4	−6	−6	−6	−4	−4	−14
13	(6+8θ)	164	0	0	−2	−4	−6	2	−2	4	−2	−2	1	−6	−2	−2	10
14	(4+9θ)	178	1	1	1	0	−3	−6	3	−4	5	8	0	−9	8	8	2
15	(12+5θ)	194	−1	0	−1	3	−6	−5	−6	−4	6	−3	−6	10	4	4	2
16	(10+7θ)	198	−1	0	−2	1	−6	0	0	−4	−4	−4	6	−6	−4	2	−10
17	(10θ)	200	0	0	0	4	4	2	2	4	4	1	−6	−6	−8	−8	2
18	(11+7θ)	219	0	0	−1	−2	0	−2	−2	−2	−6	−2	6	−6	4	−10	−2
19	(15)	225	−1	−1	−1	−4	−4	2	2	4	4	1	10	10	4	4	−14
20	(2+11θ)	246	−1	1	−3	−3	−4	1	−7	−5	1	4	−12	1	−2	1	−11
21	(7+10θ)	249	−1	−1	1	−2	−5	−1	−7	2	4	−6	9	−11	−7	11	−4

Table 5.2.5 (Concluded)

#	f	Nf	θ	1+θ	1−θ	3+θ	3−θ	3+2θ	3−2θ	1+3θ	1−3θ	5	3+4θ	3−4θ	5+3θ	5−3θ	7
22	(16)	256	0	2	−2	6	−6	6	6	−2	2	10	−6	−6	−10	10	−14
23	(16)	256	0	−2	2	−6	6	6	6	2	−2	10	−6	−6	10	−10	−14
24	(16)	256	0	−2	−2	2	2	−2	−2	−2	−2	−6	−6	−6	−6	−6	2
25	(16)	256	0	2	2	−2	−2	−2	−2	2	2	−6	−6	−6	6	6	2
26	(16+θ)	258	1	−1	2	4	−2	−6	0	−6	4	−2	0	6	−1	−2	−6
27	(16+2θ)	264	0	2	−1	2	1	−4	0	8	4	−8	2	−2	10	−12	2
28	(5+11θ)	267	0	1	−2	0	0	−6	6	2	8	2	−6	6	2	8	−10
29	(12θ)	288	0	1	1	4	4	−6	−6	−4	−4	−6	2	2	4	4	2
30	(12θ)	288	0	−1	−1	−4	−4	−6	−6	4	4	−6	2	2	−4	−4	2
31	(12θ)	288	0	−1	1	4	−4	2	2	4	−4	10	−6	−6	4	−4	2
32	(12θ)	288	0	1	−1	−4	4	2	2	−4	4	10	−6	−6	−4	4	2
33	(17)	289	−1	0	0	0	0	1	1	−4	−4	−6	−6	−6	4	4	2
34	(15+6θ)	297	0	0	1	0	−1	6	6	−4	2	2	0	12	−4	2	−10
35	(13+8θ)	297	0	−2	0	−1	3	3	−3	−7	2	8	6	12	−1	8	−4
36	(13+8θ)	297	2	2	0	−1	−5	3	5	1	−2	−4	6	0	−9	4	12

§5.3 The results of Computations for $Q(\sqrt{-3})$

Table 5.3.1	Ideals $\mathfrak{a}$ of $Z[\rho]$ with $N\mathfrak{a} \le 500$ and $\dim V(\mathfrak{a}) > 0$
Table 5.3.2	Rational Newforms in $V^+(\mathfrak{a})$
Table 5.3.3	Rational Newforms in $V^-(\mathfrak{a})$
Table 5.3.4	Elliptic Curves with small conductor
Table 5.3.5	Elliptic Curves with small conductor: Zeta Functions

Table 5.3.1 Ideals a of $Z[\rho]$ with $Na \leq 500$ and dim $V(a) > 0$

Only one ideal of each conjugate pair is listed. The splitting field is Q unless otherwise stated. Numbers in parentheses give the dimension of the appropriate subspace of newforms, when this is less than that of the whole space.

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
(7)	49	1	0	1	
(8 + ρ)	73	1	1	0	
(5 + 5ρ)	75	1	1	0	
(10)	100	1	0	1	
(11)	121	2	1	1	
(10 + 2ρ)	124	2	1	1	
(7 + 7ρ)	147	3 (1)	1	2 (0)	
(13)	169	1	0	1	
(9 + 6ρ)	171	2	1	1	
(8 + 8ρ)	192	1	1	0	
(14)	196	3 (1)	1	2 (0)	
(11 + 5ρ)	201	1	0	1	
(10 + 7ρ)	219	2 (0)	2 (0)	0	
(15)	225	3 (1)	2 (0)	1	
(14 + 2ρ)	228	1	1	0	
(15 + ρ)	241	2	1	1	
(16 + ρ)	273	1	1	0	
(11 + 8ρ)	273	1	1	0	
(13 + 6ρ)	283	1	1	0	
(17)	289	2	1	1	
(16 + 2ρ)	292	2 (0)	2 (0)	0	
(10 + 10ρ)	300	5 (1)	3 (1)	2 (0)	
(18)	324	2	1	1	
(15 + 5ρ)	325	1	0	1	
(14 + 7ρ)	343	3 (1)	1	2 (0)	
(19)	361	3	1	2	$Q(\sqrt{33})$
(11 + 11ρ)	363	5 (1)	3 (1)	2 (0)	
(14 + 8ρ)	372	4 (0)	2 (0)	2 (0)	
(15 + 7ρ)	379	1	1	0	
(17 + 5ρ)	399	1	1	0	
(13 + 10ρ)	399	2	0	2	

Table 5.3.1 (Concluded)

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
(20)	400	3 (1)	1	2 (0)	
(18 + 4ρ)	412	1	1	0	
(16 + 7ρ)	417	2	1	1	
(18 + 5ρ)	439	1	0	1	
(21)	441	7 (2)	3 (1)	4 (1)	
(13 + 12ρ)	469	2	0	2	$Q(\sqrt{13})$
(15 + 10ρ)	475	2	1	1	
(16 + 9ρ)	481	3	1	2	$Q(\sqrt{6})$
(22)	484	5 (1)	2 (0)	3 (1)	
(17 + 8ρ)	489	1	0	1	
(20 + 4ρ)	496	4 (0)	2 (0)	2 (0)	

Table 5.3.2 $Z[\rho]$: Rational Newforms in $V^{+}(a)$

a	Na	$1+\rho$	2	$1+2\rho$	$2+\rho$	$3+\rho$	$1+3\rho$	$3+2\rho$	$2+3\rho$	5	$5+\rho$	$1+5\rho$	$3+4\rho$	$4+3\rho$	$1+6\rho$	$6+\rho$	Other W
$(8+\rho)$	73	−2	−1	2	−4	2	2	2	2	2	−4	−10	2	2	2	−4	− $(8+\rho)$
$(5+5\rho)$	75	+	−3	0	0	−2	−2	4	4	−	0	0	−10	−10	4	4	
(11)	121	−1	0	−2	−2	4	4	0	0	−9	7	7	3	3	−6	−6	− (11)
$(10+2\rho)$	124	−1	+	−2	3	−1	−6	0	5	1	−	7	3	−2	4	−6	
$(7+7\rho)$	147	−	−3	+	+	−2	−2	4	4	−6	0	0	6	6	−4	−4	
$(9+6\rho)$	171	+	−1	−4	2	−4	2	−	2	−4	2	8	2	2	8	−4	
$(8+8\rho)$	192	+	−	0	0	−2	−2	−4	−4	−6	8	8	6	6	4	4	
(14)	196	−2	−	−	−	−4	−4	2	2	−10	−4	−4	2	2	8	8	
$(14+2\rho)$	228	−	+	−2	−2	−6	4	0	+	6	−8	2	−2	8	4	4	
$(15+\rho)$	241	0	1	0	−4	−2	6	−4	0	10	0	4	−2	6	−8	−4	− $(15+\rho)$
$(16+\rho)$	273	+	1	0	+	−	−2	4	−4	−6	−8	0	6	6	12	−4	
$(11+8\rho)$	273	−	−1	−	−4	−	2	−4	−4	2	8	−4	−10	2	8	8	
$(13+6\rho)$	283	−3	−3	−2	−3	0	−2	5	−4	−4	−1	−7	−3	4	4	−8	+ $(13+6\rho)$
(17)	289	0	−3	4	4	−2	−2	−4	−4	−6	4	4	−2	−2	4	4	− (17)
$(10+10\rho)$	300	−	−	−4	−4	2	2	−4	−4	−	8	8	2	2	−4	−4	
(18)	324	+	−	−1	−1	−4	−4	2	2	−1	5	5	2	2	−10	−10	
$(14+7\rho)$	343	0	1	+	−	−2	2	4	−4	2	4	−4	−2	−2	4	4	
(19)	361	−2	−4	−1	−1	−4	−4	−	−	−1	−4	−4	2	2	−1	−1	
$(11+11\rho)$	363	+	−3	4	4	−2	−2	0	0	−6	−8	−8	6	6	0	0	− (11)
$(15+7\rho)$	379	−2	−3	−5	−1	4	1	−2	−7	1	2	0	−8	−8	1	−4	+ $(15+7\rho)$
$(17+5\rho)$	399	+	1	0	−	−2	−2	+	4	2	0	0	−2	−10	−12	4	

Table 5.3.2 (Concluded)

a	Na	1+ρ	2	1+2ρ	2+1ρ	3+ρ	1+3ρ	3+2ρ	2+3ρ	5	5+ρ	1+5ρ	3+4ρ	4+3ρ	1+6ρ	6+ρ	Other W
(20)	400	-2	+	2	2	2	2	-4	-4	-	-4	-4	2	2	-10	-10	
(18+4ρ)	412	-3	+	-3	-4	1	-5	1	0	-2	2	-2	1	-4	-11	12	+ (9+2ρ)
(16+7ρ)	417	-	-2	1	1	0	0	-8	-1	-2	4	4	10	-4	-12	-5	+ (3+10ρ)
(21)	441	-	-1	-	-	2	2	-4	-4	2	-4	-4	2	2	-4	-4	
(15+10ρ)	475	0	1	4	-4	6	-2	+	-4	-	0	0	-10	-2	8	0	
(16+9ρ)	481	-2	-3	-2	0	-6	+	0	-4	-2	-2	10	+	10	-6	0	

Table 5.3.3 $\mathbb{Z}[\rho]$: Rational Newforms in $V^{-}(a)$

a	Na	1+ρ	2	1+2ρ	2+ρ	3+ρ	1+3ρ	3+2ρ	2+3ρ	5	5+ρ	1+5ρ	3+4ρ	4+3ρ	1+6ρ	6+ρ	Other W
(7)	49	0	1	+	−	−2	2	−4	4	−2	−4	4	−2	2	−4	4	
(10)	100	0	+	−2	2	4	−4	−4	4	+	−4	4	−8	8	8	−8	
(11)	121	0	3	2	−2	2	−2	−6	6	−6	4	−4	6	−6	6	−6	+ (11)
(10+2ρ)	124	−3	+	−2	−1	1	2	−4	1	1	−	−5	−5	−10	8	10	
(13)	169	0	1	−2	2	+	−	2	−2	10	2	−2	−2	2	8	−8	
(9+6ρ)	171	+	1	4	2	4	2	−	−2	4	2	−8	−2	2	8	4	
(11+5ρ)	201	+	−1	0	0	−6	−6	−4	0	−2	4	−	2	−10	8	−4	+ (2+7ρ)
(15)	225	−	3	0	0	2	−2	4	−4	+	0	0	10	−10	4	−4	
(15+ρ)	241	2	2	−3	−4	5	−6	−1	0	−1	−3	2	2	3	10	4	+ (15+ρ)
(17)	289	0	0	2	−2	5	−5	−1	1	9	4	−4	−10	10	1	−1	+ (17)
(18)	324	+	+	1	−1	4	−4	2	−2	1	5	−5	−2	2	−10	10	
(15+5ρ)	325	−2	1	2	−2	+	−2	−8	−4	−	−4	0	6	2	−6	−6	
(13+10ρ)	399	−	3	4	−	2	−6	0	−	2	0	4	10	6	4	−4	
(13+10ρ)	399	−	−1	−4	+	2	2	−8	−	−6	−8	−4	−6	−2	−4	4	
(16+7ρ)	417	−	−1	0	0	−2	2	0	−4	2	−8	4	−6	−2	4	−12	− (3+10ρ)
(18+5ρ)	439	2	1	3	−2	−7	1	−6	5	1	5	7	4	−9	6	4	− (18+5ρ)
(21)	441	−	3	−	+	2	−2	4	−4	6	0	0	−6	6	−4	4	
(15+10ρ)	475	−2	−1	2	−4	6	4	+	4	+	0	0	4	−2	2	12	
(22)	484	0	+	−2	2	−2	2	2	−2	10	−4	4	−2	2	−10	10	+ (11)
(17+8ρ)	489	+	2	4	−1	6	3	1	0	9	1	8	3	12	−8	−2−	− (3+11ρ)

Table 5.3.4 Elliptic Curves over $Q(\sqrt{-3})$ with small conductor

The following region was searched: $a_1, a_3 \in \{0, 1, \rho, \rho-1\}$; $a_2 \in \{x+y\rho : -1 \le x, y \le 1\}$; $a_4, a_6 \in \{x+y\rho : -3 \le x, y \le 3\}$.

#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies
1	0	$1+\rho$	ρ	ρ	0	$-8+3\rho$	49	CM(3) 0	$(3+5\rho)$	49	
2	1	$1+\rho$	0	ρ	0	$-1+9\rho$	73		$(8+\rho)$	73	
3	$-1+\rho$	$-\rho$	1	0	0	-15	225		$(5+5\rho)$	75	
4	0	0	1	0	0	-27	729	CM(3) 0	(9)	81	
5	0	-1	1	0	0	-11	121		(11)	121	
6	$-\rho$	$-1+\rho$	$1-\rho$	0	0	$-12+2\rho$	124		$(10+2\rho)$	124	
7	0	0	0	0	1	-432	186624	CM(3) 0	(12)	144	
8	$-1+\rho$	0	0	$-1+\rho$	0	-63	3969		$(7+7\rho)$	147	
	ρ	0	0	4ρ	-1	3969	15752961		$(7+7\rho)$	147	
9	$-1+\rho$	ρ	ρ	ρ	0	$3+21\rho$	513		$(9+6\rho)$	171	
10	0	ρ	0	$-1+\rho$	0	-48	2304		$(8+8\rho)$	192	
11	ρ	0	1	0	0	-28	784		(14)	196	
12	$-1+\rho$	$1+\rho$	1	0	$-\rho$	$252+36\rho$	73872		$(14+2\rho)$	228	
13	$-1+\rho$	$1-\rho$	ρ	0	0	$-16+15\rho$	241		$(15+\rho)$	241	
14	0	$2-\rho$	0	$1-\rho$	0	16	256	CM(3) 0	(16)	256	
15	0	$1+\rho$	0	ρ	0	16	256	CM(3) 0	(16)	256	
16	$-1+\rho$	1	$-1+\rho$	0	0	$-1+17\rho$	273		$(16+\rho)$	273	
17	1	$1+\rho$	1	$-1+\rho$	-1	$-57+33\rho$	2457		$(11+8\rho)$	273	
	$-\rho$	$1+\rho$	$1-\rho$	$5-3\rho$	-2ρ	$2160-2673\rho$	6036849		$(11+8\rho)$	273	
18	ρ	ρ	ρ	0	0	$13+6\rho$	283		$(13+6\rho)$	283	

Table 5.3.4 (Concluded)

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#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies
19	ρ	$1-\rho$	1	0	0	17	289		(17)	289	
20	1	0	1	1	2	-2160	4665600		$(10+10\rho)$	300	
21	$-1+\rho$	ρ	1	$-1+\rho$	-1	-216	46656		(18)	324	3
	ρ	$1-\rho$	0	3ρ	3	-54	2916		(18)	324	
22	0	$-\rho$	1	$-1+\rho$	0	-19	361		(19)	361	
23	$-\rho$	$1+\rho$	1	ρ	0	$7-22\rho$	379		$(15+7\rho)$	379	
24	$1-\rho$	$-1+\rho$	$1-\rho$	$-4+2\rho$	$-2+\rho$	$183-87\rho$	25137		$(17+5\rho)$	399	
25	0	$-1+\rho$	0	ρ	0	80	6400		(20)	400	
26	ρ	$1-\rho$	ρ	0	0	$-44+36\rho$	1648		$(18+4\rho)$	412	
27	0	$-\rho$	$-\rho$	-2ρ	$-1+\rho$	$-432-189\rho$	303993		$(16+7\rho)$	417	
28	$-\rho$	ρ	$-\rho$	$-2+2\rho$	$-1+\rho$	$80-105\rho$	9025		$(15+10\rho)$	475	
29	$-\rho$	$1+\rho$	0	ρ	0	$25-16\rho$	481		$(16+9\rho)$	481	
30	1	1	0	-11	0	$3^6 11^2$	$3^{12} 11^4$		$(11+11\rho)$	363	

Missing conductors: $(14+7\rho)$, (21).

Table 5.3.5 Elliptic Curves over $Q(\sqrt{-3})$: Zeta Functions

For each isogeny class of curves in Table 5.3.4, we give the Trace of Frobenius at the first fifteen primes.

	#	f	Nf	$1+\rho$	2	$1+2\rho$	$2+\rho$	$3+\rho$	$1+3\rho$	$3+2\rho$	$2+3\rho$	5	$5+\rho$	$1+5\rho$	$3+4\rho$	$4+3\rho$	$1+6\rho$	$6+\rho$
	1	$(3+5\rho)$	49	-3	-2	1	0	7	-7	-8	-1	5	4	11	-11	10	-5	-5
	2	$(8+\rho)$	73	-2	-1	2	-4	2	2	2	2	2	-4	-10	2	2	2	-4
15A-H	3	$(5+5\rho)$	75	-1	-3	0	0	-2	-2	4	4	1	0	0	-10	-10	4	4
	4	(9)	81	0	-4	-1	-1	5	5	-7	-7	-10	-4	-4	11	11	8	8
	5	(11)	121	-1	0	-2	-2	4	4	0	0	-9	7	7	3	3	-6	-6
	6	$(10+2\rho)$	124	-1	-1	-2	3	-1	-6	0	5	1	1	7	3	-2	4	-6
	7	(12)	144	0	0	-4	-4	2	2	8	8	-10	-4	-4	-10	-10	8	8
21A-F	8	$(7+7\rho)$	147	1	-3	-1	-1	-2	-2	4	4	-6	0	0	6	6	-4	-4
	9	$(9+6\rho)$	171	0	-1	-4	2	-4	2	1	2	-4	2	8	2	2	8	-4
24A-F	10	$(8+8\rho)$	192	-1	0	0	0	-2	-2	-4	-4	-6	8	8	6	6	4	4
14A-F	11	(14)	196	-2	1	1	1	-4	-4	2	2	-10	-4	-4	2	2	8	8
	12	$(14+2\rho)$	228	1	-1	-2	-2	-6	4	0	-1	6	-8	2	-2	8	4	4
	13	$(15+\rho)$	241	0	1	0	-4	-2	6	-4	0	10	0	4	-2	6	-8	-4
	14	(16)	256	0	0	4	-4	-2	-2	-8	8	10	-4	4	-10	-10	8	-8
	15	(16)	256	0	0	-4	4	-2	-2	8	-8	10	4	-4	-10	-10	-8	8
	16	$(16+\rho)$	273	-1	1	0	-1	1	-2	4	-4	-6	-8	0	6	6	12	-4
	17	$(11+8\rho)$	273	1	-1	1	-4	1	2	-4	-4	2	8	-4	-10	2	8	8
	18	$(13+6\rho)$	283	-3	-3	-2	-3	0	-2	5	-4	-4	-1	-7	-3	4	4	-8
17A-F	19	(17)	289	0	-3	4	4	-2	-2	-4	-4	-6	4	4	-2	-2	4	4
	20	$(10+10\rho)$	300	1	1	-4	-4	2	2	-4	-4	1	8	8	2	2	-4	-4
54A-F	21	(18)	324	0	1	-1	-1	-4	-4	2	2	-1	5	5	2	2	-10	-10

Table 5.3.5 (Concluded)

19A-E
20A-D

#	f	Nf	1+ρ	2	1+2ρ	2+ρ	3+ρ	1+3ρ	3+2ρ	2+3ρ	5	5+ρ	1+5ρ	3+4ρ	4+3ρ	1+6ρ	6+ρ
22	(19)	361	-2	-4	-1	-1	-4	-4	1	1	-1	-4	-4	2	2	-1	-1
23	(15+7ρ)	379	-2	-3	-5	-1	4	1	-2	-7	1	2	0	-8	-8	1	-4
24	(17+5ρ)	399	-1	1	0	1	-2	-2	-1	4	2	0	0	-2	-10	-12	4
25	(20)	400	-2	0	2	2	2	2	-4	-4	1	-4	-4	2	2	-10	-10
26	(18+4ρ)	412	-3	-1	-3	-4	1	-5	1	0	-2	2	-2	1	-4	-11	12
27	(16+7ρ)	417	1	-2	1	1	0	0	-8	-1	-2	4	4	10	-4	-12	-5
28	(15+10ρ)	475	0	1	4	-4	6	-2	-1	-4	1	0	0	-10	-2	8	0
29	(16+9ρ)	481	-2	-3	-2	0	-6	-1	0	-4	-2	-2	10	-1	10	-6	0
30	(11+11ρ)	363	-1	-3	4	4	-2	-2	0	0	-6	-8	-8	6	6	0	0

§5.4 The Results of Computations for $Q(\sqrt{-7})$

Table 5.4.1	Ideals a of $Z[\frac{1}{2}(1+\sqrt{-7})]$ with $Na \le 200$ and $\dim V(a) > 0$
Table 5.4.2	Rational Newforms in $V^{+}(a)$
Table 5.4.3	Rational Newforms in $V^{-}(a)$
Table 5.4.4	Elliptic Curves with small conductor
Table 5.4.5	Elliptic Curves with small conductor: Zeta Functions

Table 5.4.1 Ideals a of $Z[\frac{1}{2}(1 + \sqrt{-7})]$ with Na $\leq$ 200 and dim V(a) > 0

Only one ideal of each conjugate pair is listed. The splitting field is Q unless otherwise stated. Numbers in parentheses give the dimension of the appropriate subspace of newforms, when this is less than that of the whole space.

a	Na	dim V(a)	dim V^+(a)	dim V^-(a)	Splitting Field
(5)	25	1	0	1	
(-2 + 4α)	28	1	1	0	
(6)	36	1	0	1	
(2 + 4α)	44	1	1	0	
(-1 + 5α)	46	1	1	0	
(5α)	50	2 (0)	0	2 (0)	
(6 + 2α)	56	2 (0)	2 (0)	0	
(-3 + 6α)	63	1	1	0	
(8)	64	1	0	1	
(6α)	72	2 (0)	0	2 (0)	
(1 + 6α)	79	1	0	1	
(4 + 5α)	86	1	1	0	
(-3 + 7α)	86	1	1	0	
(8 + 2α)	88	2 (0)	2 (0)	0	
(7 + 3α)	88	1	1	0	
(2 + 6α)	88	3 (1)	3 (1)	0	
(-2 + 7α)	88	1	1	0	
(9 + α)	92	2 (0)	2 (0)	0	
(6 + 4α)	92	2 (0)	2 (0)	0	
(10)	100	5 (1)	0	5 (1)	
(5 + 5α)	100	3 (0)	0	3 (0)	
(8 + 3α)	106	1	0	1	
(4 + 6α)	112	4 (1)	3 (0)	1	
(-4 + 8α)	112	4 (0)	4 (0)	0	
(2 + 7α)	116	1	0	1	
(-2 + 8α)	116	1	0	1	
(11)	121	3	1	2	$Q(\sqrt{17})$
(9 + 3α)	126	2 (0)	2 (0)	0	
(8α)	128	3 (1)	1	2 (0)	
(12)	144	5 (1)	0	5 (1)	
(6 + 6α)	144	4 (1)	0	4 (1)	

Table 5.4.1 (concluded)

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
$(8 + 5\alpha)$	154	1	0	1	
$(-1 + 9\alpha)$	154	1	0	1	
$(12 + \alpha)$	158	3 (1)	1	2 (0)	
$(5 + 7\alpha)$	158	2 (0)	0	2 (0)	
$(3 + 8\alpha)$	161	1	1	0	
$(7 + 6\alpha)$	163	1	0	1	
(13)	169	5	0	5	$Q(\sqrt{3}), Q(x^3+4x^2-2)$
$(11 + 3\alpha)$	172	2 (0)	2 (0)	0	
$(10 + 4\alpha)$	172	4 (0)	4 (0)	0	
$(1 + 9\alpha)$	172	2 (0)	2 (0)	0	
$(-5 + 10\alpha)$	175	5 (3)	3	2 (0)	Q, $Q(\sqrt{17})$
$(12 + 2\alpha)$	176	8 (1)	7 (0)	1	
$(9 + 5\alpha)$	176	3 (1)	2 (0)	1	
$(6 + 7\alpha)$	176	3 (1)	2 (0)	1	
$(4 + 8\alpha)$	176	6 (0)	6 (0)	0	
$(-4 + 10\alpha)$	176	5 (0)	5 (0)	0	
$(8 + 6\alpha)$	184	4 (1)	3 (0)	1	
$(2 + 9\alpha)$	184	3 (0)	3 (0)	0	
$(-2 + 10\alpha)$	184	6 (2)	5 (1)	1	
(14)	196	5 (3)	3 (1)	2	
$(3 + 9\alpha)$	198	1	0	1	
$(10 + 5\alpha)$	200	5 (1)	1	4 (0)	
(10α)	200	8 (0)	0	8 (0)	

Table 5.4.2 $Z[\frac{1}{2}(1 + \sqrt{-7})]$: Rational Newforms in $V^+(a)$

a	Na	α	$1-\alpha$	$1-2\alpha$	3	$1+2\alpha$	$3-2\alpha$	$3+2\alpha$	$5-2\alpha$	5	$-1+4\alpha$	$3-4\alpha$	$1+4\alpha$	$5-4\alpha$	$5+2\alpha$	$7-2\alpha$	Other W
$(-2+4\alpha)$	28	+	+	−	−2	0	0	0	0	−10	−6	−6	2	2	8	8	
$(2+4\alpha)$	44	+	−	−4	−2	+	0	0	0	2	6	6	2	−10	−4	−4	
$(-1+5\alpha)$	46	−1	−	0	2	−4	−4	−8	+	2	6	−2	−2	6	4	−4	
$(-3+6\alpha)$	63	−1	−1	+	−	4	4	0	0	−6	−2	−2	6	6	−4	−4	
$(4+5\alpha)$	86	+	0	−1	4	3	3	−6	−3	−1	6	−6	−7	−7	8	−	
$(-3+7\alpha)$	86	1	−	−4	2	−4	4	0	0	−2	−6	−2	6	−2	0	+	
$(7+3\alpha)$	88	1	−	4	−2	+	0	−8	0	−6	6	−2	2	−2	4	−4	
$(2+6\alpha)$	88	+	−	2	4	−6	+	0	0	2	6	−6	2	−4	−4	−4	
$(-2+7\alpha)$	88	−	−1	0	2	+	4	8	0	−6	−2	−10	6	−2	−4	4	
(11)	121	−2	−2	−2	−5	−	−	−1	−1	−9	0	0	3	3	−6	−6	
(8α)	128	+	−	0	2	−4	4	0	0	2	−2	−2	−10	−10	4	−4	
$(12+\alpha)$	158	+	−2	−1	−1	−6	−3	1	−8	−3	3	−4	10	4	2	−8	+ $(7-6\alpha)$
$(3+8\alpha)$	161	0	0	−	4	−3	6	3	+	−1	−6	3	−7	2	−1	8	
$(5-10\alpha)$	175	0	0	−	−5	−3	−3	−6	−6	−	3	3	2	2	−10	−10	
$(-2+10\alpha)$	184	+	−	−4	4	6	0	0	+	−4	6	0	2	2	−4	8	
(14)	196	−	−	−	−4	−2	−2	−4	−4	−2	2	2	10	10	2	2	
$(10+5\alpha)$	200	+	−2	−4	−1	−3	−1	−8	−4	+	9	−3	4	−8	−6	2	

Table 5.4.3 $Z[\frac{1}{2}(1+\sqrt{-7})]$: Rational Newforms in $V^{-}(a)$

a	Na	α	$1-\alpha$	$1-2\alpha$	3	$1+2\alpha$	$3-2\alpha$	$3+2\alpha$	$5-2\alpha$	5	$-1+4\alpha$	$3-4\alpha$	$1+4\alpha$	$5-4\alpha$	$5+2\alpha$	$7-2\alpha$	Other W
(5)	25	-2	-2	0	-3	1	1	-4	-4	+	-1	-1	0	0	6	6	
(6)	36	+	+	0	+	-4	-4	0	0	-6	2	2	6	6	-4	-4	
(8)	64	+	+	0	-2	-4	-4	0	0	2	2	2	-10	-10	4	4	
$(1+6\alpha)$	79	-2	1	-3	-3	-5	-2	-4	-1	1	8	5	0	-6	0	6	$-(1+6\alpha)$
(10)	100	-	-	0	2	-4	-4	-4	-4	+	-6	-6	10	10	-4	-4	
$(8+3\alpha)$	106	+	-1	0	2	-4	0	4	-8	-2	2	2	-10	10	0	-4	$+(7-4\alpha)$
$(4+6\alpha)$	112	+	+	+	-2	4	4	0	-8	-6	-6	2	-2	6	-4	-4	
$(2+7\alpha)$	116	-	-1	2	-2	2	0	-6	6	-6	6	-	-6	10	-6	-8	
$(-2+8\alpha)$	116	+	-	0	2	-4	0	4	-4	-6	-	-10	-2	-10	4	12	
(12)	144	-	+	0	+	2	2	-6	-6	6	-10	-10	-6	-6	-4	-4	
$(6+6\alpha)$	144	-	-	4	-	4	-4	-4	-4	-2	-2	-6	-2	-2	-12	-4	
$(8+5\alpha)$	154	+	-1	-	2	-	-4	-4	4	-2	-6	10	6	-2	8	-4	
$(-1+9\alpha)$	154	1	+	-	-2	+	-4	0	8	-6	2	-6	-10	-2	-12	-4	
$(7+6\alpha)$	163	-2	1	3	-3	4	-2	-4	5	-8	-1	-4	6	-3	6	-9	$-(7+6\alpha)$
$(12+2\alpha)$	176	-	+	4	2	+	0	0	-8	2	10	2	-6	-2	12	-12	
$(9+5\alpha)$	176	1	-	0	-2	-4	+	0	-8	-6	10	2	-2	6	-4	-4	
$(6+7\alpha)$	176	+	-1	4	2	0	-	0	-8	-6	2	-6	-2	2	-4	-4	
$(8+6\alpha)$	184	-	+	2	0	4	0	4	-	-4	0	10	4	-6	-10	-12	
$(-2+10\alpha)$	184	-	+	-4	0	-2	0	-8	-	8	-6	4	10	-6	-4	0	
(14)	196	-	-	+	4	-2	-2	-4	-4	2	2	2	-10	-10	-2	-2	
(14)	196	+	+	+	2	0	0	0	0	10	-6	-6	-2	-2	-8	-8	
$(3+9\alpha)$	198	-1	-	0	-	0	+	4	-8	2	-2	-6	-6	-2	-8	4	

Table 5.4.4 Elliptic Curves over $\mathbb{Q}(\sqrt{-7})$ with small conductor

Search region: $a_1, a_3 \in \{0, 1, \alpha, \alpha-1\}$; $a_2 \in \{x+y\alpha : -1 \le x, y \le 1\}$; $a_4, a_6 \in \{x+y\alpha : -3 \le x \le 2, -2 \le y \le 3\}$.

	#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies
	1	α	$2-\alpha$	α	$2-2\alpha$	$1-\alpha$	$-46+45\alpha$	4096	CM(7) -15^3	$(2-3\alpha)$	16	
14A	2	1	0	1	-1	0	-28	784		$(2-4\alpha)$	28	
		1	α	$-1+\alpha$	1	α	$2+10\alpha$	224		$(2-4\alpha)$	28	
	3	1	$1-\alpha$	0	1	1	$84-316\alpha$	180244		$(2+4\alpha)$	44	
		1	1	$-\alpha$	$2-\alpha$	$2-\alpha$	$368-144\alpha$	123904		$(2+4\alpha)$	44	
	4	1	$-\alpha$	$-\alpha$	$-2+\alpha$	1	$73-43\alpha$	5888		$(1-5\alpha)$	46	
		0	$-1+\alpha$	$-1+\alpha$	0	0	$79+19\alpha$	8464		$(1-5\alpha)$	46	
	5	1	-1	0	-2	-1	-343	117649	CM(7) -15^3	(7)	49	
	6	1	$1-\alpha$	$-1+\alpha$	α	0	$-3+6\alpha$	63		$(3-6\alpha)$	63	
21A		1	0	0	1	0	-63	3969		$(3-6\alpha)$	63	
	7	$-1+\alpha$	$2-\alpha$	1	-1	0	$-38+17\alpha$	1376		$(4+5\alpha)$	86	
	8	1	$2-\alpha$	1	$-\alpha$	-1	$-89+7\alpha$	7396		$(3-7\alpha)$	86	
		1	$1+\alpha$	0	$2+\alpha$	1	$5+17\alpha$	688		$(3-7\alpha)$	86	
		1	$-\alpha$	1	$-1-2\alpha$	$-3+2\alpha$	$3-7\alpha$	86		$(3-7\alpha)$	86	
	9	$-1+\alpha$	α	$-1+\alpha$	α	1	$-337+195\alpha$	123904		$(7+3\alpha)$	88	
	10	$1-\alpha$	0	$1-\alpha$	$-1+\alpha$	α	$448-64\alpha$	180224		$(2+6\alpha)$	88	
		$-1+\alpha$	$-1+\alpha$	0	$-3+2\alpha$	$-1+\alpha$	$120+8\alpha$	15448		$(2+6\alpha)$	88	
	11	$-\alpha$	$-1+\alpha$	0	1	0	$-78-35\alpha$	11264		$(2-7\alpha)$	88	
		$-\alpha$	α	$-\alpha$	$-\alpha$	$1-\alpha$	$146-115\alpha$	30976		$(2-7\alpha)$	88	
11A	12	0	-1	1	0	0	-11	121		(11)	121	

Table 5.4.4 (Concluded)

#	a_1	a_2	a_3	a_4	a_6	Δ	$N\Delta$	CM & j	f	Nf	Isogenies
13	0	$1-\alpha$	0	$-2+\alpha$	0	$-256-256\alpha$	262144		(8α)	128	
	0	$-\alpha$	0	$-1-\alpha$	0	$-512+256\alpha$	262144		(8α)	128	
	0	$-1+\alpha$	0	$-4+2\alpha$	$-2+\alpha$	$160-112\alpha$	32768		(8α)	128	
	0	α	0	$-2-\alpha$	$1+\alpha$	$48+112\alpha$	32768		(8α)	128	
14	$1-\alpha$	$-\alpha$	1	$-1+\alpha$	0	$-18-15\alpha$	1264		$(12+\alpha)$	158	
15	0	$2-\alpha$	1	$-\alpha$	-1	$-3-8\alpha$	161		$(3+8\alpha)$	161	
16	0	1	1	-1	0	-35	1225		$(5-10\alpha)$	175	
17	$1-\alpha$	$2-\alpha$	$1-\alpha$	$2-\alpha$	$-\alpha$	$60-116\alpha$	23552		$(2-10\alpha)$	184	
	$1-\alpha$	α	$1-\alpha$	2	0	$38-98\alpha$	16928		$(2-10\alpha)$	184	
18	0	$1-\alpha$	$-\alpha$	-1	0	$-70-15\alpha$	6400		$(10+5\alpha)$	200	

35A

Missing conductor: (14).

Table 5.4.5 Elliptic Curves over Q(√-7): Zeta Functions

For each isogeny class of curves in Table 5.4.4, we give the Trace of Frobenius at each of the first fifteen primes.

#	f	Nf	α	1−α	1−2α	3	1+2α	3−2α	3+2α	5−2α	5	−1+4α	3−4α	1+4α	5−4α	5+2α	7−2α
1	(2−3α)	16	0	−1	0	−6	4	−4	−8	8	10	−2	−2	6	6	12	−12
2	(2−4α)	28	−1	−1	1	−2	0	0	0	0	−10	−6	−6	2	2	8	8
3	(2+4α)	44	−1	1	−4	−2	−1	0	0	0	2	6	6	2	−10	−4	−4
4	(1−5α)	46	−1	1	0	2	−4	−4	−8	−1	2	6	−2	−2	6	4	−4
5	(7)	49	1	1	0	−6	4	4	8	8	−10	2	2	−6	−6	−12	−12
6	(3−6α)	63	−1	−1	−1	1	4	4	0	0	−6	−2	−2	6	6	−4	−4
7	(4+5α)	86	−1	0	−1	4	3	3	−6	−3	−1	6	−6	−7	−7	8	1
8	(3−7α)	86	1	1	−4	2	−4	4	0	0	−2	−6	−2	6	−2	0	−1
9	(7+3α)	88	1	0	4	−2	−1	0	−8	0	−6	6	−2	2	−2	4	−4
10	(2+6α)	88	−1	0	2	4	−6	−1	0	0	2	6	−6	2	−4	−4	−4
11	(2−7α)	88	0	−1	0	2	−1	4	8	0	−6	−2	−10	6	−2	−4	4
12	(11)	121	−2	−2	−2	−5	1	1	−1	−1	−9	0	0	3	3	−6	−6
13	(8α)	128	0	0	0	2	−4	4	0	0	2	−2	−2	−10	−10	4	−4
14	(12+α)	158	−1	−2	−1	−1	−6	−3	1	−8	−3	3	−4	10	4	2	−8
15	(3+8α)	161	0	0	1	4	−3	6	3	−1	−1	−6	3	−7	2	−1	8
16	(5−10α)	175	0	0	1	−5	−3	−3	−6	−6	1	3	3	2	2	−10	−10
17	(2−10α)	184	−1	0	−4	4	6	0	0	−1	−4	6	0	2	2	−4	8
18	(10+5α)	200	0	−2	−4	−1	−3	−1	−8	−4	−1	9	−3	4	−8	−6	2

§5.5 The Results of Computations for $\mathbb{Q}(\sqrt{-11})$

Table 5.5.1 Ideals a of $Z[\frac{1}{2}(1 + \sqrt{-11})]$ with $Na \leq 200$ and $\dim V(a) > 0$

Only one ideal of each conjugate pair is listed. The splitting field is Q unless otherwise stated. Numbers in parentheses give the dimension of the appropriate space of newforms, when this is less than that of the whole space.

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
(3)	9	1	0	1	
$(-1 + 2\alpha)$	11	1	1	0	
(5)	25	1	0	1	
(3α)	27	3 (1)	1	2 (0)	
$(5 + \alpha)$	33	2 (0)	2 (0)	0	
(6)	36	3 (1)	0	3 (1)	
$(-2 + 4\alpha)$	44	2 (0)	2 (0)	0	
$(3 + 3\alpha)$	45	2 (0)	0	2 (0)	
$(5 + 2\alpha)$	47	1	1	0	
(7)	49	3	0	3	$Q(x^3+x^2-8x-4)$
$(4 + 3\alpha)$	55	2 (0)	2 (0)	0	
(8)	64	1	0	1	
$(-2 + 5\alpha)$	69	2	0	2	
$(-1 + 5\alpha)$	71	2	0	2	$Q(\sqrt{5})$
$(7 + 2\alpha)$	75	2	0	2	
(5α)	75	2 (0)	0	2 (0)	
(9)	81	9 (1)	4 (0)	5 (1)	
$(6 + 3\alpha)$	81	5 (0)	2 (0)	3 (0)	
$(2 + 5\alpha)$	89	2	1	1	
$(8 + 2\alpha)$	92	1	1	0	
$(3 + 5\alpha)$	99	6 (3)	4 (1)	2	
$(-3 + 6\alpha)$	99	7 (1)	5 (1)	2 (0)	
(10)	100	5 (3)	0	5 (3)	
$(6 + 4\alpha)$	108	4	2	2	
(6α)	108	9 (1)	3 (1)	6 (0)	
$(9 + 2\alpha)$	111	4	2	2	$Q(\sqrt{5}), Q(\sqrt{5})$
(11)	121	5 (3)	3 (1)	2	
$(5 + 5\alpha)$	125	3 (1)	0	3 (1)	
$(10 + 2\alpha)$	132	4 (0)	4 (0)	0	
$(9 + 3\alpha)$	135	6 (0)	2 (0)	4 (0)	
$(3 + 6\alpha)$	135	8 (2)	3 (1)	5 (1)	

Table 5.5.1 (Concluded)

a	Na	dim V(a)	dim $V^+(a)$	dim $V^-(a)$	Splitting Field
$(6 + 5\alpha)$	141	3 (1)	2 (0)	1	
$(-1 + 7\alpha)$	141	2 (0)	2 (0)	0	
(12)	144	5 (0)	0	5 (0)	
(7α)	147	6 (0)	0	6 (0)	
$(9 + 4\alpha)$	165	5 (1)	4 (0)	1	
$(2 + 7\alpha)$	165	8 (4)	7 (3)	1	$Q(\sqrt{5})$
(13)	169	5	0	5	$Q(x^5-x^4-11x^3+7x^2+17x+7)$
$(-4 + 8\alpha)$	176	4 (1)	4 (1)	0	
$(3 + 7\alpha)$	177	1	1	0	
$(6 + 6\alpha)$	180	7 (1)	1	6 (0)	
$(-1 + 8\alpha)$	185	3	3	0	$Q(x^3-4x-2)$
$(10 + 4\alpha)$	188	2 (0)	2 (0)	0	
(8α)	192	2 (0)	0	2 (0)	
(14)	196	9 (3)	1	8 (2)	$Q(\sqrt{3})$

Table 5.5.2 $Z[\frac{1}{2}(1+\sqrt{-11})]$: Rational Newforms in $V^+(a)$

a	Na	2	α	1−α	1+α	2−α	−1+2α	4+α	5−α	1+3α	4−3α	2+3α	5−3α	5+2α	7−2α	7	Other W
(−1+2α)	11	0	−1	−1	1	1	−	−1	−1	7	7	3	3	8	8	−10	
(3α)	27	1	−	+	−2	2	0	4	−4	0	0	−2	−2	8	−8	−6	
(5+2α)	47	0	1	−3	−3	−1	−4	−3	3	5	−5	7	7	+	0	2	
(2+5α)	89	−1	0	2	2	2	−4	0	6	4	−2	−6	2	−8	12	10	− (2+5α)
(8+2α)	92	−	−2	1	0	3	3	+	0	−4	5	−7	2	−6	3	5	
(3+5α)	99	3	+	2	−2	−2	−	−4	8	−2	4	−12	0	−4	2	8	
(−3+6α)	99	−3	+	+	−2	−2	−	8	8	−8	−8	6	6	8	8	2	
(6+4α)	108	−	−	−2	0	−3	−3	−6	−6	5	−4	−7	−7	0	3	−4	
(6+4α)	108	−	+	1	0	3	−6	3	0	−4	5	2	11	−6	−6	−4	
(6α)	108	−	−	−	0	0	0	−6	6	−4	−4	2	2	6	−6	14	
(11)	121	−3	2	2	1	1	−	2	2	−2	−2	−3	−3	2	2	−10	
(3+6α)	135	−1	−	+	0	+	6	6	6	−4	8	−10	2	0	0	2	
(2+7α)	165	3	−2	+	−	4	+	0	−6	−4	0	−2	−8	−12	4	0	
(−4+8α)	176	+	1	1	−3	−3	+	−3	−3	5	5	−1	−1	0	0	−10	
(3+7α)	177	−2	−	0	−3	0	2	−5	−7	−3	4	−8	−6	2	8	−5	− (8−α)
(6+6α)	180	−	+	+	+	2	4	−4	4	−8	8	−2	−2	4	12	2	
(14)	196	−	−2	−2	0	0	0	0	0	−4	−4	2	2	−12	−12	−	

Table 5.5.3 $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-11})]$: Rational newforms in $V^{-}(a)$

a	Na	2	α	$1-\alpha$	$1+\alpha$	$2-\alpha$	$-1+2\alpha$	$4+\alpha$	$5-\alpha$	$1+3\alpha$	$4-3\alpha$	$2+3\alpha$	$5-3\alpha$	$5+2\alpha$	$7-2\alpha$	7	Other W
(3)	9	-1	+	+	-2	-2	0	4	4	0	0	2	2	-8	-8	-6	
(5)	25	1	-2	-2	+	+	0	-6	-6	-4	-4	10	10	6	6	2	
(6)	36	+	-	-	0	0	0	-6	-6	-4	-4	-2	-2	-6	-6	14	
(8)	64	+	-1	-1	-1	-1	0	3	3	-9	-9	-5	-5	-4	-4	-2	
$(-2+5\alpha)$	69	-1	1	+	3	-2	-5	-6	-	-5	0	2	-3	-3	12	9	
$(-2+5\alpha)$	69	3	-3	-	-1	-2	3	-6	-	3	-8	2	1	9	4	-11	
$(7+2\alpha)$	75	-1	1	+	+	3	0	-6	-6	10	-5	-8	2	-3	-3	4	
$(7+2\alpha)$	75	1	1	-	-	-3	0	6	-6	-10	5	-8	-2	3	-3	-4	
(9)	81	-1	-	+	2	2	0	-4	-4	0	0	2	2	8	8	-6	
$(2+5\alpha)$	89	-2	-3	2	4	2	-1	-3	-6	4	-2	3	4	7	-9	-2	$-\ (2+5\alpha)$
$(3+5\alpha)$	99	0	-	-1	1	-1	+	-1	1	7	7	-3	-3	-8	8	-10	
$(3+5\alpha)$	99	-3	+	2	-2	2	+	-4	-8	-2	4	12	0	4	2	8	
(10)	100	+	1	1	+	+	0	0	0	5	5	-11	-11	-6	-6	5	
(10)	100	+	3	3	-	-	0	-4	-4	1	1	3	3	-6	-6	-11	
(10)	100	+	-2	-2	-	-	0	6	6	-4	-4	-2	-2	-6	-6	14	
$(6+4\alpha)$	108	+	-	1	0	-3	6	3	0	-4	5	-2	-11	6	-6	-4	
$(6+4\alpha)$	108	+	-	-2	0	3	3	-6	6	5	-4	7	7	0	3	-4	
(11)	121	0	1	1	-1	-1	-	1	1	-7	-7	3	3	-8	-8	10	
(11)	121	3	-2	-2	-1	-1	-	-2	-2	2	2	-3	-3	-2	-2	10	
$(5+5\alpha)$	125	-1	-2	2	+	-	0	6	-6	4	4	10	-10	-6	6	-2	
$(3+6\alpha)$	135	1	-	-	0	+	6	-6	6	-4	8	10	-2	0	0	2	

Table 5.5.3 (Concluded)

a	Na	2	α	$1-\alpha$	$1+\alpha$	$2-\alpha$	$-1+2\alpha$	$4+\alpha$	$5-\alpha$	$1+3\alpha$	$4-3\alpha$	$2+3\alpha$	$5-3\alpha$	$5+2\alpha$	$7-2\alpha$	7	Other W
$(6+5\alpha)$	141	1	-	0	-1	-1	1	4	-3	-5	2	6	10	-3	-	-1	
$(9+4\alpha)$	165	-1	-	0	-	2	-	-3	8	-4	0	-2	-2	0	-4	6	
$(2+7\alpha)$	165	1	-2	-	+	0	+	0	6	-4	-4	-2	-8	0	12	8	

Table 5.5.4 Elliptic Curves over Q($\sqrt{-11}$) with small conductor

Search region: $a_1, a_3 \in \{0, 1, \alpha, \alpha-1\}$; $a_2 \in \{x+y\alpha : -1 \le x, y \le 1\}$; $a_4, a_6 \in \{x+y\alpha : -2 \le x, y \le 2\}$.

	#	a_1	a_2	a_3	a_4	a_6	Δ	NΔ	CM & j	f	Nf	Isogenies
11A	1	0	-1	1	0	0	-11	121		$(-1+2\alpha)$	11	
	2	$-\alpha$	α	$1-\alpha$	$-\alpha$	$2-\alpha$	$729-243\alpha$	531441		(3α)	27	
	3	0	$1-\alpha$	1	-1	0	$13+24\alpha$	2209		$(5+2\alpha)$	47	
	4	$-\alpha$	α	0	-1	0	$-2-5\alpha$	89		$(2+5\alpha)$	89	
	5	$1-\alpha$	α	1	-1	0	$-16-4\alpha$	368		$(8+2\alpha)$	92	
	6	1	$2-\alpha$	0	-2α	-1	$-339-136\alpha$	216513		$(3+5\alpha)$	99	
	7	$-\alpha$	0	1	α	0	$-30-2\alpha$	972		$(6+4\alpha)$	108	3
		1	$-1+\alpha$	α	$-\alpha$	1	$24+16\alpha$	1728		$(6+4\alpha)$	108	
	8	$1-\alpha$	$2-\alpha$	1	1	0	$-24-16\alpha$	1728		$(6+4\alpha)$	108	
	9	α	$1+\alpha$	α	α	0	$-27+81\alpha$	18225		$(3+6\alpha)$	135	
	10	0	$1-\alpha$	α	α	0	$-618+623\alpha$	1161297		$(3+7\alpha)$	177	
14A	11	1	0	1	-1	0	-28	784		(14)	196	
33B	12	1	1	0	-11	0	$3^6 11^2$	$3^{12} 11^4$		$(3-6\alpha)$	99	
121F	13	1	1	0	-2	-7	-11^4	11^8		(11)	121	
44A	14	0	1	0	3	-1	$-2^8 11$	$2^{16} 11^2$		$(4-8\alpha)$	176	

Missing conductors: $(\cancel{6\alpha})$, $(2\cancel{+7\alpha})$, $(6\cancel{+6\alpha})$.

Table 5.5.5   Elliptic Curves over $Q(\sqrt{-11})$: Zeta Functions

For each isogeny class of curves in Table 5.5.4, we give the Trace of Frobenius at the first fifteen primes.

#	f	Nf	2	α	$1-\alpha$	$1+\alpha$	$2-\alpha$	$-1+2\alpha$	$4+\alpha$	$5-\alpha$	$1+3\alpha$	$4-3\alpha$	$2+3\alpha$	$5-3\alpha$	$5+2\alpha$	$7-2\alpha$	7
1	$(-1+2\alpha)$	11	0	-1	-1	1	1	1	-1	-1	7	7	3	3	8	8	-10
2	(3α)	27	1	0	-1	-2	2	0	4	-4	0	0	-2	-2	8	-8	-6
3	$(5+2\alpha)$	47	0	1	-3	-3	-1	-4	-3	3	5	-5	7	7	-1	0	2
4	$(2+5\alpha)$	89	-1	0	2	2	2	-4	0	6	4	-2	-6	2	-8	12	10
5	$(8+2\alpha)$	92	1	-2	1	0	3	3	-1	0	-4	5	-7	2	-6	3	5
6	$(3+5\alpha)$	99	3	0	2	-2	-2	1	-4	8	-2	4	-12	0	-4	2	8
7	$(6+4\alpha)$	108	1	0	1	0	3	-6	3	0	-4	5	2	11	-6	-6	-4
8	$(6+4\alpha)$	108	1	0	-2	0	-3	-3	-6	-6	5	-4	-7	-7	0	3	-4
9	$(3+6\alpha)$	135	-1	1	0	0	-1	6	6	6	-4	8	-10	2	0	0	2
10	$(3+7\alpha)$	177	-2	1	0	-3	0	2	-5	-7	-3	4	-8	-6	2	8	-5
11	(14)	196	1	-2	-2	0	0	0	0	0	-4	-4	2	2	-12	-12	1
12	$(3-6\alpha)$	99	-3	-1	-1	-2	-2	1	8	8	-8	-8	6	6	8	8	2
13	(11)	121	-3	2	2	1	1	0	2	2	-2	-2	-3	-3	2	2	-10
14	$(4-8\alpha)$	176	0	1	1	-3	-3	-1	-3	-3	5	5	-1	-1	0	0	-10

§5.6 Some comments on the results in the tables

Consider the tables of results for $Q(\sqrt{-1})$, recorded in the tables in §5.1. They support the following claim.

Claim A (i) For every newform in $V^+(a)$ there corresponds an isogeny class of elliptic curves defined over $Q(\sqrt{-1})$ with conductor a;

(ii) For primes p not dividing a, the Trace of Frobenius of the curve at p is equal to the eigenvalue of T_p acting on the space generated by the newform;

(iii) For primes p dividing a: if p^2 divides a then the Trace of Frobenius of the curve at p is 0, otherwise (if p divides a exactly) it is minus the corresponding eigenvalue of W_p;

(iv) Every elliptic curve defined over $Q(\sqrt{-1})$ corresponds to a newform in $V^+(a)$ in this way, where a is the conductor of the curve, except when the curve has complex multiplication by an order in $Q(\sqrt{-1})$.

Parts (i), (ii) and (iii) of Claim A also hold for the other four fields, as one can verify by inspecting the appropriate tables. We are unable to prove them, however. Part (iv) has an obvious analogue for the other four fields: curves defined over K with complex multiplication by an order in K do not correspond to cusp forms. The reason for this is that the zeta function of such a curve is known to be a Hecke L-series with Grossencharacter; one can attach automorphic forms to such objects, but they are not cusp forms. (c.f. [7] Theorem 2(b).)

The main motivation for the work which went into the computations, the results of which are recorded in §5.1 - §5.5 was to be able to state precisely a conjecture relating automorphic forms for congruence subgroups of $GL(2,K)$ with elliptic curves defined over K. That such a connection exists is suggested not only by the well known results for the case when the ground field is Q, but also by the general philosophy relating automorphic forms to L-series which satisfy a functional equation, via a generalization of the Mellin transform. However, although this very general approach certainly predicts a connection between elliptic curves defined over a complex quadratic field and the cusp forms of weight 2 discussed in Chapter 3, it is hard to extract from it the precise nature of the connection. We hope that our claim does this to some extent.

In [11] and [12], Mennicke and Grunewald also discuss this question. Their computations of newforms at prime level give some evidence for Claim A, and they also remark that, as in part (iv) of the claim, one would not expect a cusp form to correspond to an elliptic curve with complex multiplication by the ground field K. They also suggest that in certain cases a newform may exist in $V^+(\mathfrak{a})$ for some ideal $\mathfrak{a}$ without a corresponding curve of conductor $\mathfrak{a}$, but our results do not seem to support this: however, it is possible that the situation at a low level is not typical. Of course, it would be desirable to have a procedure for constructing an elliptic curve directly from a newform f(z), as Tingley did in the rational case by means of calculating the periods of the differential $2\pi i f(z)$; but all efforts in this direction have so far been unsuccessful. Recall that in the rational case, $X_N(\mathbb{C})$ has, as well as a complex structure, an algebraic structure as an algebraic curve $X_0(N)$, and that elliptic curves arise as one-dimensional factors of the Jacobian $J_0(N)$. By contrast we have (apparently) no complex or algebraic structure on $\Delta_0(\mathfrak{a})\backslash H_3^*$ by means of which to generalize this construction. We have tried calculating the periods of the differential corresponding to a newform, integrated around a corresponding pair of cycles in V^+ and V^-, for some of the newforms of §5.1: but the numbers which result have no obvious interpretation in terms of the expected elliptic curve. The results of these integrations are recorded in the last section of this Chapter.

According to claim A, it is only the newforms in $V^+(\mathfrak{a})$ which are related to elliptic curves. However, there is a connection between V^+ and V^-. We state this first as a second claim, for the case $K = \mathbf{Q}(\sqrt{-1})$.

<u>Claim B</u> There is a one-one correspondance between newforms in V^+ and newforms in V^-, not necessarily at the same level: if the levels are $\mathfrak{a}_1$ and $\mathfrak{a}_2$ resectively, then <u>either</u> $\mathfrak{a}_1 = \mathfrak{a}_2 \cap (1+i)^4$ <u>or</u> $\mathfrak{a}_2 = \mathfrak{a}_1 \cap (1+i)^4$.

To make the correspondance in Claim B clearer we give some examples. There is a newform in $V^+((6+6i))$ which grows into two oldforms in $V^+((12))$; in $V^-((12))$ there is a newform. These two newforms have the same eigenvalue for T_π if $\pi \equiv 1 \pmod 2$, and hence eigenvalues of opposite sign for T_π if $\pi \equiv i \pmod 2$, by (3.3.6). Note that $(12) = (6+6i) \cap (1+i)^4$. In the other direction, there is a newform in $V^-((8+2i))$, which grows into two oldforms in $V^-((10+6i))$ and three oldforms in $V^-((16+4i))$; now $(16+4i) = (8+2i) \cap (1+i)^4$, and there is a newform in $V^+((16+4i))$ whose eigenvalues correspond as before.

One other example: Mennicke in [11] observed that there is a newform in $V^-((11+4i))$, but (apparently) no elliptic curve with conductor $(11+4i)$. Having calculated $V^\pm(\mathfrak{a})$ for $\mathfrak{a} = (11+4i)$, $(11+4i)(1+i)$, $(11+4i)(1+i)^2$, $(11+4i)(1+i)^3$ and $(11+4i)(1+i)^4$, we eventually find, as well as five oldforms in $V^-((11+4i)(1+i)^4)$, a newform in $V^+((11+4i)(1+i)^4)$ as predicted by Claim B. Moreover there is an elliptic curve with conductor $(11+4i)(1+i)^4$ (which Mennicke had found), whose Traces of Frobenius correspond as in Claim A.

We can rephrase Claim B as follows.

<u>Theorem</u> *Let* $\mathfrak{a}$ *be an ideal of* $\mathbb{Z}[i]$ *such that* $(1+i)^4$ *divides* $\mathfrak{a}$. *Then there is a map* $R_2 : V(\mathfrak{a}) \to V(\mathfrak{a})$ *such that*

(i) $R_2 J = -JR_2$;

(ii) $R_2 T_\pi = \left(\frac{\pi}{2}\right) T_\pi R_2$ if π is prime, $(\pi) \neq (1+i)$, and $(\pi) \nmid \mathfrak{a}$;

(iii) $R_2 W_\pi = \left(\frac{\pi}{2}\right) W_\pi R_2$ if π is prime, $(\pi) \neq (1+i)$, and $(\pi) \mid \mathfrak{a}$.

Here $\left(\frac{\pi}{2}\right)$ denotes the quadratic character modulo (2): it is +1 if $\pi \equiv 1$

(mod 2), and -1 if $\pi \equiv i \pmod 2$. It follows from (i) that R_2 maps $V^+(\mathfrak{a})$ into $V^-(\mathfrak{a})$ and vice versa; then from (ii) and (iii) it follows that R_2 preserves the eigenvalue of T_π and W_π provided that $\pi \equiv 1 \pmod 2$. Note that every prime ideal of $Z[i]$ has four generators, of which two are congruent to 1 and two congruent to i (mod 2). So we can always choose a generator in such a way that R_2 preserves eigenvalues. This was done in the Tables of §5.1 in order to make the correspondence between newforms in V^+ and V^- more striking.

This Theorem will be proved in the next chapter as an application of a more general result. The map R_2 is a special case of a whole class of 'twisting' operators, one for each ideal $\mathfrak{q}$ of $Z[i]$, which acts on $V(\mathfrak{a})$ provided that $\mathfrak{q}^2$ divides $\mathfrak{a}$ (here $\mathfrak{q}$ is either (2) or an odd prime of $Z[i]$).

Similarly, for the other four fields, these twisting operators can be defined whenever the 'square of the twist' divides the level. Other examples in the Tables: for $Q(\sqrt{-2})$ there are examples of $(1+\theta)$-twists, for example at level $(2+5\theta)$ where there are corresponding newforms simultaneously in V^+ and V^- (note that $(1+\theta)^2$ divides $(2+5\theta)$). There are (2)-twists at level (12), and (4)-twists at level (16). For $Q(\sqrt{-3})$ the twists visible in the tables are the $(1+\rho)$-twist, at level (15) for example, and a $(2+\rho)$-twist at level $(14+7\rho)$. In general, a $\mathfrak{q}$-twist will preserve or reverse the J eigenvalue according as ε is or is not a square modulo $\mathfrak{q}$, where ε generates the unit group $\mathcal{O}_K^*$ of $\mathcal{O}_K$.

Twisting operators will be developed and studied more systematically and in detail in the next Chapter: their existence and basic properties are readily suggested by the tables of results above. As a matter of history, the map R_2 which interchanges $V^+(\mathfrak{a})$ and $V^-(\mathfrak{a})$, provided that (4) divides $\mathfrak{a}$, was discovered by the author as a means of proving Claim B. It was only later, while trying to extend the result to other fields, that the twisting operators of Atkin-Lehner were remembered; and then it became a straightforward matter of generalizing the results of Atkin and

Lehner in the last section of [3] to produce a general theory of twisting operators, as developed in the next Chapter.

§5.7 Some Calculations of Periods

We refer to Section 3.5, and in particular to formula (3.5.3). For a few selected newforms for $\Delta_0(a)$, where a is an ideal of Z[i], we have carried out the following computations: first we determine an element g of $\Delta_0(a)$ such that the image of the path $\{\infty, g\infty\}$ in $H_1(\Delta_0(a)\backslash H_3^*, Q)$ generates the corresponding eigenspace. Of the possible choices for g we find one with lower left-hand entry as small as possible, for reasons of convergence as explained in §3.5. Then we calculate a large number of Hecke eigenvalues for the newform: in practice we computed the eigenvalue of T_π for all primes π with norm less than 500. Next we calculate the coefficients $c(\xi)$ of the newform, for $\xi \in Z[i]$, given the multiplicativity and recurrence relations of §3.3: for powers of primes which do not divide a we use (3.3.9)(i) and (ii); if π divides a but π^2 does not, we set $c(\pi)$ to be minus the eigenvalue of W_π, and $c(\pi^r) = c(\pi)^r$; and if π^2 divides a we set $c(\pi^r) = 0$ for $r \geq 1$.

Now we can substitute in (3.5.3). We used a numerical (polynomial) approximation to evaluate K_1 (c.f. [1] §9.8). To save time, we calculated together the four terms corresponding to associate integers ξ: note that in (3.5.3) the only factor, apart from $c(\xi)$, which depends on ξ itself rather than just $|\xi|$ is $\psi(\eta^{-1}\alpha\xi) = \exp(-\pi i(\alpha\xi + \overline{\alpha\xi}))$ (since the different η is 2 here). Denote by $\Sigma_\varepsilon(f;\gamma)$ the expression on the right hand side of (3.5.3) with the sum restricted to those $\xi \in Z[i]$ with $\xi \equiv \varepsilon \pmod{(2+2i)}$, for $\varepsilon = 1, i, -1$ and $-i$. Then clearly

$$\Sigma_1 = \overline{\Sigma}_{-1} \qquad \text{and} \qquad \Sigma_i = \overline{\Sigma}_{-i}$$

since $\psi(-z) = \overline{\psi(z)}$.

Example 1 $a = (12+4i)$

Here $(1+i)^4$ divides a so that the (2)-twist operates: there is a newform in each of V^+ and V^-, with corresponding eigenvalues (equal for $\pi \equiv 1 \pmod 2$, opposite for $\pi \equiv i \pmod 2$). $V^+((12+4i))$ is generated by M-symbol $(3+2i:1)$ which corresponds to a path $\{P,\gamma_+P\}$ for any point P, with

$$\gamma_+ = \begin{pmatrix} 7-2i & 1 \\ 2(12+4i) & 3+2i \end{pmatrix} .$$

Secondly, $V^-((12+4i))$ is generated by M-symbol $(3i:1)$ corresponding to $\{P,\gamma_-P\}$ for

$$\gamma_- = \begin{pmatrix} 4+i & 1 \\ i(12+4i) & 3i \end{pmatrix} .$$

Let F^+ be the form with coefficients from the eigenvalues of V^+, and F^- the form from V^-. Then

$$\Sigma_1(F^\pm,\gamma_+) \approx 5.28\pi(1+i)/\sqrt{(160)}, \text{ and}$$

$$\Sigma_i(F^+,\gamma_+) \approx 5.28\pi(1+i)/\sqrt{(160)}, \text{ while}$$

$$\Sigma_1(F^\pm,\gamma_-) \approx 5.28\pi(-1+i)/\sqrt{(160)}, \text{ and}$$

$$\Sigma_i(F^+,\gamma_-) \approx 5.28\pi(1+i)/\sqrt{(160)}.$$

<u>Example 2</u> <u>$a = (1+i)^9$</u>

a) There is a pair of newforms in V^+ and V^- corresponding to the curve IX2 of §5.1. Setting F to be the newform in S^+ we have calculated

$$\Sigma_1(F,\gamma_+) \approx (7.06 + 10.02i)\pi/8\sqrt{2},$$

$$\Sigma_i(F,\gamma_+) \approx (7.06 - 2.77i)\pi/8\sqrt{2},$$

$$\Sigma_1(F,\gamma_-) \approx (7.05 + 10.00i)\pi/8\sqrt{2},$$

and $$\Sigma_i(F,\gamma_-) \approx (-7.05 - 11.33i)\pi/8\sqrt{2}.$$

b) There is also a pair of newforms in V^+ and V^- corresponding to the curve IX1 of §5.1. Setting F to be the corresponding newform in S^+, we have

$$\Sigma_1(F,\gamma_+) \approx (-0.00 + 2.97i)\pi/8\sqrt{2},$$

$$\Sigma_i(F,\gamma_+) \approx (-0.01 + 9.84i)\pi/8\sqrt{2},$$

$$\Sigma_1(F,\gamma_-) \approx (\ 0.00 - 8.07i)\pi/8\sqrt{2},$$

and $$\Sigma_i(F,\gamma_-) \approx (-0.00 - 9.84i)\pi/8\sqrt{2}.$$

<u>Example 3 $\underline{a}_1 = (8+2i)$, $\underline{a}_2 = (16+4i)$</u>

There is a newform F_1 in $V^-((8+2i))$ whose (2)-twist is a newform F_2 in $V^+((16+4i))$. Denote the corresponding cycles by γ_- and γ_+. We have

$$\Sigma_1(F_1,\gamma_-) \approx (-4.82 + 3.09i)\pi/2\sqrt{68},$$

$$\Sigma_i(F_1,\gamma_-) \approx (3.66 - 0.41i)\pi/2\sqrt{68},$$

$$\Sigma_1(F_2,\gamma_+) \approx (8.48 + 1.17i)\pi/2\sqrt{68},$$

and

$$\Sigma_i(F_2,\gamma_+) \approx (8.48 + 8.48i)\pi/2\sqrt{68}.$$

CHAPTER 6

Twisting Operators

In this Chapter, we define certain operators on the spaces $S(\mathfrak{a})$ of cusp forms for $\Delta_0(\mathfrak{a})$ defined in Chapter 3. Most generally, whenever we have an ideal $\mathfrak{b}$ with a 'quadratic' character $\chi : (O_K/\mathfrak{b})^\times \to \{\pm 1\}$, then we will be able to define an operator R_χ on $S(\mathfrak{a})$ provided that $\mathfrak{b}^2$ divides $\mathfrak{a}$. Under suitable conditions, such R_χ will interchange the two eigenspaces for the main involution J; their effect on Hecke and W eigenspaces will be determined; and in certain cases the twisting operators will enable us to construct newforms from oldforms. The motivation of this work was to prove Claim B of §5.6, by finding an explicit connection between V^+ and V^- over $\mathbb{Q}(i)$: newforms in V^- did not seem to correspond to elliptic curves directly, but there was always a related newform in V^+, possibly at a different level, which did have a corresponding elliptic curve.

These twisting operators are discussed fully in the rational case by Atkin and Lehner in Section 6 of [3]. The characters χ which are used there are either the quadratic character modulo an odd prime q, or the characters modulo 4 and 8. In order to determine all quadratic characters for a complex quadratic field K we have to determine the structure of $(O_K/\mathfrak{b})^\times$ for an arbitrary ideal $\mathfrak{b}$ of O_K. This will be done in the first section. In the second section, the twisting operators will be defined and their properties developed. In the third section, we will prove a result, analogous to Atkin-Lehner's Theorem 6 for cusp forms over $\mathbb{Q}$, showing how certain newforms arise as twists of oldforms. Lastly, in the fourth section, we illustrate, with examples taken from the results of Chapter 5, how the connection between the V^+ and V^- spaces of newforms arises by means of certain twists.

§6.1 Quadratic Characters of O_K

By a quadratic character of O_K we will mean a surjective homomorphism

$$\chi : (O_K/b)^\times \to \{\pm 1\}$$

extended to O_K as follows: for $x \in O_K$, relatively prime to the ideal b, set $\chi(x)$ equal to $\chi(\bar{x})$, where $\bar{x}$ is the reduction of x modulo b; otherwise set $\chi(x) = 0$. The largest ideal b modulo which χ is defined (that is, the ideal of smallest norm) will be called the conductor of χ.

It is clear that, in order to determine all such characters, it suffices to determine the characters modulo prime powers, since an arbitrary quadratic character will be a product of these, in the obvious way, by the Chinese Remainder Theorem. For powers of an odd prime p (that is, one not containing the number 2), this is achieved by the following lemma.

Lemma 6.1.1 *If p is an odd prime ideal of O_K, there are no quadratic characters with conductor p^e unless e = 1; there is a unique character with conductor p given by the quadratic residue symbol.*

Proof: $(O_K/p^e)^\times \simeq G_1 \times G_2$ where G_1 has order $N(p)^{e-1}$ (which is odd) and consists of the residues modulo p^e which are congruent to 1 modulo p; while G_2 maps isomorphically onto $(O_K/p)^\times$ under reduction modulo p, so is cyclic of order $N(p)-1$, being the multiplicative group of the finite field O_K/p. A quadratic character must clearly be trivial on G_1, while on G_2 its kernel must be the (unique) subgroup of index 2, namely the subgroup of squares.

For even primes the situation is more complicated. We need to determine the structure of $(O_K/p^e)^\times$ where p divides (2).

Case 1: Z[i] Here the unique prime dividing (2) is (1+i).

Proposition 6.1.2 If $k \le 3$ then $(Z[i]/(1+i)^k)^\times$ is cyclic of order 2^{k-1}, generated by i. If $k \ge 3$ then

$$(Z[i]/(1+i)^k)^\times \simeq \langle i\rangle \times \langle 5\rangle \times \langle -1+2i\rangle$$

where

(i) i *has order* 4;

(ii) if $k = 2n$ *is even, then* 5 *has order* 2^{n-2} *and* $-1+2i$ *has order* 2^{n-1};

(iii) if $k = 2n-1$ *is odd, then* 5 *and* $-1+2i$ *both have order* 2^{n-1}.

Proof: This is a fairly straightforward matter of verifying that the elements given have the orders stated, and that the subgroups they generate have trivial intersection.

From the proposition, it follows that there are essentially three quadratic characters with conductor $(1+i)^k$ for some k:

<u>k = 2</u> $\chi_1(x) = -1$ if $x \equiv i \pmod 2$;

$+1$ if $x \equiv 1 \pmod 2$.

<u>k = 4</u> $\chi_2(x) = +1$ if $x \equiv \pm 1, \pm i \pmod 4$;

-1 if $x \equiv \pm 1+2i,\ 2\pm i \pmod 4$.

<u>k = 5</u> $\chi_3(x) = +1$ if $x \equiv \pm 1, \pm i, \pm(1-2i), \pm(2+i) \pmod{4+4i}$;

-1 otherwise.

Ofcourse, $\chi_1\chi_2$ is another character with conductor $(1+i)^4$, and $\chi_1\chi_3$, $\chi_2\chi_3$, and $\chi_1\chi_2\chi_3$ are other characters with conductor $(1+i)^5$.

Characters χ_2 and χ_3 are related to quadratic residues, in the following way. First note that both χ_2 and χ_3 have value +1 at i, so they can be defined on <u>ideals</u> of $Z[i]$ as well as on elements. It turns out that, for an odd prime p of $Z[i]$,

(i) $\chi_2(p) = \left(\frac{i}{p}\right) = \left(\frac{2}{p}\right)$;

(ii) $\chi_3(p) = \left(\frac{1+i}{p}\right)$; where as usual the symbol $\left(\frac{\alpha}{p}\right)$ is +1 if α is a square modulo p, and -1 otherwise. This is analogous to the situation in Z, where the unique quadratic character of conductor 4 is $p \to \left(\frac{-1}{p}\right)$, while the quadratic characters of conductor 8 are $p \to \left(\frac{2}{p}\right)$ and $p \to \left(\frac{-2}{p}\right)$.

For the sake of brevity we do not describe all the quadratic characters of even prime power conductor for the other Euclidean fields, but just give the characters of smallest conductor.

<u>Case 2: $Z[\sqrt{-2}]$</u> As usual, we set $\theta = \sqrt{-2}$; the unique prime dividing (2) is (θ). There is a character χ_1 of conductor $(\theta)^2 = (2)$, namely:

$\chi_1(x) \;=\; +1 \quad \text{if } x \equiv 1 \pmod 2;$

$\qquad\quad\; -1 \quad \text{if } x \equiv 1+\theta \pmod 2.$

Since $\chi_1(-1) = 1$, we may define χ_1 on ideals of $Z[\theta]$. A simple check shows that for an odd prime ideal p,

$$\chi_1(p) \;=\; \left(\frac{-1}{p}\right) \;=\; \left(\frac{2}{p}\right).$$

Also, $Z[\theta]/(\theta)^{3\,4} = \langle -1\rangle \times \langle 1+\theta\rangle$, $\simeq C_2 \times C_4$ ~~a Klein four-group~~. So there is a character χ_2 with kernel $\langle 1+\theta\rangle$, and $\chi_1\chi_2$ has kernel $\langle -1-\theta\rangle$. There is also a character χ_3 with conductor $(\theta)^5$ such that $\chi_3(-1) = 1$ and $\chi_3(p) = \left(\frac{\theta}{p}\right)$ for odd primes p.

<u>Case 3:</u> <u>$Z[\rho]$, $\rho = \frac{1}{2}(1+\sqrt{-3})$</u> Write $R = Z[\rho]$ for short. The ideal (2) is prime in R, so that $(R/(2))^\times$ is cyclic of order 3, generated by ρ; so there are no quadratic characters with conductor (2). However,

$$(R/(2)^2)^\times \;=\; \langle\rho\rangle \times \langle 1+2\rho\rangle$$

where ρ has order 6 and $1+2\rho$ has order 2; so there are three quadratic characters with conductor $(2)^2$: χ_1, with kernel $\langle\rho\rangle$; χ_2, with kernel $\langle\rho^2\rangle \times \langle 1+2\rho\rangle$; and $\chi_3 = \chi_1\chi_2$.

Notice that since χ_1 is trivial on the units of R, it can be defined on ideals of R: in fact we have, for an odd prime ideal p:

$$\chi_1(p) \;=\; \left(\frac{-1}{p}\right) \;=\; \left(\frac{\rho}{p}\right).$$

[Proof: The second inequality holds trivially since $-\rho$ is a square in R. Also, $\left(\frac{\rho}{p}\right) = 1$ if and only if R/p has an element of order 12, which is if and only if $Np \equiv 1 \pmod{12}$. This is true for an inert prime p, if p has the form (p) with $p \in Z$ and $p \equiv -1 \pmod 3$, and these are in the kernel of χ_1. For split primes p, of the form $(a+b\rho)$, with $Np = a^2+ab+b^2 = p \equiv 1 \pmod 3$, a simple check shows that $p \equiv 1 \pmod 4$ if and only if $a+b\rho$ is in the kernel of χ_1.]

<u>Case 4:</u> <u>$Z[\alpha]$, $\alpha = \frac{1}{2}(1+\sqrt{-7})$</u> In this case (2) splits into two distinct prime ideals: $(2) = (\alpha)(\bar{\alpha})$. We have

$$Z[\alpha]/(\alpha)^k \;\simeq\; Z/(2^k) \;\simeq\; Z[\alpha]/(\bar{\alpha})^k.$$

From our knowledge of quadratic characters of Z modulo a power of 2, we deduce that:

(i) There is a character χ_1 modulo $(\alpha)^2$, namely $\chi_1(1) = 1$, $\chi_1(-1) = -1$; similarly there is a character $\overline{\chi}_1$ modulo $(\overline{\alpha})^2$.

(ii) There are two characters modulo $(\alpha)^3$: first, χ_2, defined by $\chi_2(1) = \chi_2(-1) = 1$, $\chi_2(3) = \chi_2(-3) = -1$; and then $\chi_3 = \chi_2\chi_1$. Similarly, there are characters $\overline{\chi}_2$ and $\overline{\chi}_3$ modulo $(\overline{\alpha})^3$.

(iii) There are no more characters modulo powers of (α) or $(\overline{\alpha})$.

A simple verification shows that $\chi_2(p) = \left(\frac{-\alpha}{p}\right)$.

Case 5: $Z[\alpha]$, $\alpha = \frac{1}{2}(1+\sqrt{-11})$ In this case (2) remains prime, so there are no quadratic characters of $(Z[\alpha]/(2))$, since this has order 3. But $(Z[\alpha]/(2)^2) = \langle -1\rangle \times \langle\alpha\rangle$, where α has order 6, so there are three quadratic characters with conductor $(2)^2$:

χ_1 with kernel $\langle\alpha\rangle$;

χ_2 with kernel $\langle -1\rangle \times \langle\alpha^2\rangle$;

$\chi_3 = \chi_1\chi_2$.

A simple verification shows that $\chi_2(p) = \left(\frac{-1}{p}\right)$.

§6.2 Twisting Operators: Definition and Elementary Properties

Let $\mathfrak{q}$ be an ideal of O_K with a quadratic character

$$\chi : (O_K/\mathfrak{q})^\times \to \{\pm 1\}.$$

Assume that $\mathfrak{q}$ is principal, generated by an element q of O_K. Let $R_q = \begin{pmatrix} q & 1 \\ 0 & q \end{pmatrix}$; then $R_q^\lambda = \begin{pmatrix} q & \lambda \\ 0 & q \end{pmatrix}$ (in PGL(2)) for natural numbers λ. We extend this definition to arbitrary $\lambda \in O_K$ by setting

(6.2.1) $$R_q^\lambda := \begin{pmatrix} q & \lambda \\ 0 & q \end{pmatrix};$$

observe that $R_q^1 = R_q$, that $R_q^q = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and that the law of exponents holds: $R_q^\lambda R_q^\mu = R_q^{\lambda+\mu}$. Also, if $\lambda \equiv \mu \pmod q$ then $R_q^\lambda \equiv R_q^\mu \pmod{\Delta_0(\mathfrak{a})}$.

Now let $\mathfrak{a}$ be an ideal such that $\mathfrak{q}^2$ divides $\mathfrak{a}$. If $\gamma = \begin{pmatrix} a & b \\ q^2c & d \end{pmatrix}$ is an element of $\Delta_0(\mathfrak{a})$, and λ and μ are in O_K, we have

$$R_q^{\lambda}\gamma R_q^{-\mu} = \begin{pmatrix} a+\lambda qc & b+\dfrac{(\lambda d-\mu a)}{q} - \lambda\mu c \\ q^2c & d-\mu qc \end{pmatrix}$$

which is in $\Delta_0(\mathfrak{a})$ provided that $\lambda d \equiv \mu a \pmod{q}$. Since $ad-bcq^2 = 1$, we have $ad \equiv 1 \pmod{q}$, so an equivalent condition is that

(6.2.2) $$\mu \equiv \lambda d^2 \pmod{q}.$$

So for a given $\gamma \in \Delta_0(\mathfrak{a})$ and a given λ, there is a μ (unique modulo q) such that $R_q^{\lambda}\gamma R_q^{-\mu} \in \Delta_0(\mathfrak{a})$.

Fix a set $U(q) \subset O_K$ which forms a complete set of invertible residues modulo q. Then we define R_χ to be a particular element of the group algebra of $PGL(2,K)$:

(6.2.3) $$R_\chi := \sum_{\lambda\in U(q)} \chi(\lambda)R_q^{\lambda}.$$

<u>Lemma 6.2.4</u> (c.f. [3] Lemma 29) *Let* q *and* χ *be as above, and* $\mathfrak{a}$ *an ideal divisible by* q^2. *Let* $F \in S(\mathfrak{a})$, *as defined in* §3.3. *Then*

(i) $F|R_\chi$ *is in* $S(\mathfrak{a})$;

(ii) *If* $(O_K/q)^{\times}$ *has exponent 2, then for any* λ, *the matrix* R_q^{λ} *normalizes* $\Delta_0(\mathfrak{a})$, *and* $F|R_q^{\lambda}$ *is in* $S(\mathfrak{a})$.

Proof: As remarked in the proof of Lemma 3.3.7, we only have to verify that $F|R_\chi$ (respectively $F|R_q^{\lambda}$) are invariant under $\Delta_0(\mathfrak{a})$. If $\gamma \in \Delta_0(\mathfrak{a})$, then

$$(F|R_\chi)|\gamma = F|\sum_{\lambda\in U(q)} \chi(\lambda)R_q^{\lambda}\gamma = F|\sum_{\mu\in U(q)} \chi(\mu)R_q^{\mu} = F|R_\chi$$

since for a fixed γ, different λ in $U(q)$ give rise to unique distinct μ in $U(q)$, and $\chi(\mu) = \chi(\lambda d^2) = \chi(\lambda)\chi(d)^2 = \chi(\lambda)$. In (ii) the condition on q implies that $d^2 \equiv 1 \pmod{q}$ for any d relatively prime to q, so that $\mu \equiv \lambda \pmod{q}$ by (6.2.2); hence R_q^{λ} normalizes $\Delta_0(\mathfrak{a})$ as required.

<u>Lemma 6.2.5</u> *Let* q, χ, *and* $\mathfrak{a}$ *be as in the previous Lemma. Let* p *be a prime dividing* $\mathfrak{a}$ *to the exact power* e, *but not dividing* q, *generated by* $\pi \in O_K$. *Then if* $F \in S(\mathfrak{a})$ *we have*

(6.2.6) $$(F|R_\chi)|W_\pi = \chi(\pi^e)(F|W_\pi)|R_\chi.$$

Proof: Only trivial changes are needed to the proof of [3] Lemma 30.

<u>Lemma 6.2.7</u> *Let χ be a quadratic character with prime conductor* $\mathfrak{p} = (\pi)$, *and let* $\mathfrak{a}$ *be an ideal such that* $\mathfrak{p}^2$ *divides* $\mathfrak{a}$ *but* $(\mathfrak{p}^{-2}\mathfrak{a},\mathfrak{p}) = (1)$. *Then for each* λ *with* $(\lambda,\pi) = 1$ *there exists* λ' *such that*

(i) $\chi(\lambda) = \chi(\lambda')$;

(ii) $R_\pi^\lambda W_\pi R_\pi^{\lambda'} \in \Delta_0(\mathfrak{p}^{-1}\mathfrak{a})$.

Proof: Identical to [3] Lemma 31 (ii).

Recall, from 3.2.12 and 3.2.17, that a cusp form for $\Delta_0(\mathfrak{a})$ has a Fourier series expansion of the form (3.2.13);

$$(6.2.8) \qquad F(z,t) = \sum_{\alpha\in O_K} c(\alpha)H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha z)$$

where $\eta \in O_K$ generates the different δ of K, and ψ is the additive character of K, given by $\psi(z) = \exp(-2\pi i(z+\bar{z}))$.

Let $\chi : (O_K/\mathfrak{q})^\times \to \{z \in \mathbf{C} : |z| = 1\}$ be any character modulo an ideal $\mathfrak{q}$ of O_K. Extend χ to the whole of O_K by defining $\chi(\lambda) = 0$ whenever $(\lambda) + \mathfrak{q} \neq O_K$. Then we can define an operator $\bar{R}_\chi$ directly on Fourier series: if F is given by (6.2.8), then we define

$$(6.2.9) \qquad (F|\bar{R}_\chi)(z,t) := \sum_{\alpha\in O_K} \chi(\alpha)c(\alpha)H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha z).$$

<u>Proposition 6.2.10</u> *Let χ be a character with conductor* $\mathfrak{q} = (q)$, *as above, and F a harmonic function with Fourier series (6.2.8). Then*

$$F|R_\chi = g(\chi).F|\bar{R}_\chi$$

where $$g(\chi) := \sum_{\lambda \bmod q} \chi(\lambda)\psi(\eta^{-1}\lambda/q).$$

Proof: By (3.2.4) we have

$$(F|R_q^\lambda)(z,t) = F(z + \frac{\lambda}{q}, t) = \sum_\alpha c(\alpha)H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha(z + \frac{\lambda}{q}))$$
$$= \sum_\alpha c(\alpha)H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha z)\psi(\eta^{-1}\alpha\frac{\lambda}{q}).$$

Hence
$$(F|R_\chi)(z,t) = \sum_\alpha c(\alpha)H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha z) \sum_{\lambda \bmod q} \chi(\lambda)\psi(\eta^{-1}\alpha\frac{\lambda}{q})$$
$$= \sum_\alpha \chi(\alpha)c(\alpha)H(\eta^{-1}\alpha t)\psi(\eta^{-1}\alpha z) \sum_{\lambda \bmod q} \chi(\alpha\lambda)\psi(\eta^{-1}\frac{\lambda\alpha}{q})$$
$$= g(\chi).(F|\bar{R}_\chi)(z,t)$$

since if $(\alpha,q) = 1$ then

$$\sum_{\lambda \bmod q} \chi(\alpha\lambda)\psi(\eta^{-1}\frac{\alpha\lambda}{q}) = \sum_{\lambda \bmod q} \chi(\lambda)\psi(\eta^{-1}\frac{\lambda}{q}) = g(\chi),$$

while if $(\alpha,q) \neq (1)$ then $\chi(\alpha) = 0$ and

$$\sum_{\lambda \bmod q} \chi(\lambda)\psi(\eta^{-1}\frac{\alpha\lambda}{q}) = 0.$$

Proposition 6.2.11 *Under the hypotheses of the previous Proposition, suppose also that* $F \in S(\mathfrak{a})$ *where* $\mathfrak{a}$ *is divisible by* $\mathfrak{q}^2$. *Let* $\mathfrak{p}_1 = (\pi_1)$ *be a prime not dividing* $\mathfrak{a}$, *and* $\mathfrak{p}_2 = (\pi_2)$ *a prime dividing* $\mathfrak{a}$ *to the exact power* e. *Then*

(i) $(F|\overline{R}_\chi)|T_{\pi_1} = \chi(\pi_1).(F|T_{\pi_1})|\overline{R}_\chi;$

(ii) $(F|\overline{R}_\chi)|W_{\pi_2} = \chi(\pi_2^e).(F|W_{\pi_2})|\overline{R}_\chi$ *if* $\chi(\pi_2) \neq 0;$

(iii) $(F|R_\chi)|J = \chi(\varepsilon).(F|J)|R_\chi$, *where as usual* $\langle\varepsilon\rangle = O_K^*$ *and* $J=\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}$.

Proof: Part (ii) follows from Lemma 6.2.5 and Proposition 6.2.11. Part (i) follows from inspection of the Fourier series, using (6.2.9) and (3.3.7): the coefficient of $(F|\overline{R}_\chi)|T_\pi$ at $\alpha \in O_K$ is

$$\left(N(\pi)\chi(\alpha\pi)c(\alpha\pi) + \chi(\alpha/\pi)c(\alpha/\pi)\right)g(\chi)$$
$$= \chi(\pi)\chi(\alpha)\left(N(\pi)c(\alpha\pi) + c(\alpha/\pi)\right)g(\chi)$$

since $\chi(\alpha\pi^{-1}) = \chi(\pi)\chi(\alpha)$ if $\pi|\alpha$; on the other hand, the coefficient of $(F|T_\pi)|\overline{R}_\chi$ at α is

$$\chi(\alpha)\left(N(\pi)c(\alpha\pi) + c(\alpha/\pi)\right)g(\chi).$$

As for part (iii), first note that $R_q^\lambda J = JR_q^{\varepsilon^{-1}\lambda}$. Then

$$\begin{aligned}(F|R_\chi)|J &= \sum_{\lambda \bmod q} \chi(\lambda)F|R_q^\lambda J\\ &= \Sigma\, \chi(\lambda)F|JR_q^{\varepsilon^{-1}\lambda}\\ &= \chi(\varepsilon)\, \Sigma\, \chi(\varepsilon^{-1}\lambda)(F|J)|R_q^{\varepsilon^{-1}\lambda}\\ &= \chi(\varepsilon)(F|J)|R_\chi \quad \text{as required.}\end{aligned}$$

§6.3 Twisting Operators: Newforms from Old

For primes π dividing the ideal a, there is an operator U_π defined on $S(a)$ which is similar in some respects to T_π for π not dividing a. If F is a harmonic function, define

(6.3.1) $$F|U_\pi := \sum_{\lambda \bmod \pi} F\left|\begin{pmatrix} 1 & \lambda \\ 0 & \pi \end{pmatrix}\right.$$

so that $(F|U_\pi)(z,t) = \sum_{\lambda \bmod \pi} F(\frac{z+\lambda}{\pi}, \frac{t}{\pi})$. A calculation similar to the derivation of (3.3.7) shows that if F has a Fourier series with coefficients $\{c(\alpha)\}$, then $F|U_\pi$ has Fourier coefficients $\{c(\alpha\pi)\}$.

Lemma 6.3.2 *If* $F \in S(a)$ *and* π^2 *divides* a, *then* $F|U_\pi \in S(a(\pi)^{-1})$.

Proof: As for [3], Lemma 7.

Lemma 6.3.3 *If* $(\pi_1) \neq (\pi_2)$ *then* $(F|T_{\pi_1})|U_{\pi_2} = (F|U_{\pi_2})|T_{\pi_1}$.

Proof: Let F have Fourier coefficients $\{c(\alpha)\}$. Then by (3.3.7), $F|T_{\pi_1}$ has coefficients $\{N(\pi_1)c(\alpha\pi_1) + c(\alpha\pi_1^{-1})\}$, and so $(F|T_{\pi_1})|U_{\pi_2}$ has coefficients $\{N(\pi_1)c(\alpha\pi_1\pi_2) + c(\alpha\pi_2\pi_1^{-1})\}$. Applying the operators in the opposite order yields the same coefficients. Note that our convention that $c(\beta) = 0$ whenever $\beta \notin O_K$, yields

$$c((\alpha/\pi_1)\pi_2) = c((\alpha\pi_2)/\pi_1),$$

as $\pi_1|\alpha$ if and only if $\pi_1|\alpha\pi_2$, since $(\pi_1,\pi_2) = (1)$.

Lemma 6.3.4 *Let* $F \in S(a)$ *where* a *is divisible by* q^2 *and let* χ *be a quadratic character with conductor* q. *Then for any* π *dividing* a,

$$(F|R_\chi)|U_\pi = \chi(\pi)(F|U_\pi)|R_\chi.$$

Proof: Immediate from the Fourier series. Note that the result is valid even if $\chi(\pi) = 0$, for then the left-hand side is zero also.

Lemma 6.3.5 *If* F *is a newform in* $S(a)$, *and* π^2 *divides* a, *then* $F|U_\pi = 0$.

Proof: Since F is a newform it is an eigenform for all the $T_{\pi'}$, for $\pi' \nmid a$, with eigenvalues $a(\pi')$, say. Then by Lemma 6.3.3 we have

$$(F|U_\pi)|T_{\pi'} = (F|T_{\pi'})|U_\pi = a(\pi')F|U_\pi$$

and so $F|U_\pi$, which is in $S(a(\pi)^{-1})$ by Lemma 6.3.2, has the same eigenvalues as F for all $\pi' \nmid a$. Since $F|U_\pi$ is an oldform in $S(a)$, it follows from the results stated at the end of §3.3 that $F|U_\pi = 0$.

Proposition 6.3.6 *Let* G *be a newform for* $\Delta_0(\mathrm{b})$ *for some ideal* b *dividing* a, *and let* G^* *be a member of the corresponding 'oldclass' for* $\Delta_0(\mathrm{a})$*: that is,* G^* *is a linear combination of functions* $G\left|\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\right.$ *for various divisors* λ *of* ab^{-1}. *If* π *is a prime dividing* a *such that* G^* *is an eigenform of both* U_π *and* W_π *then* π *does not divide* ab^{-1}.

The proof of the analogous proposition in the rational case is in [3], as part of the proof of Theorem 5 there. As it is fairly long and entirely technical we omit it here.

Corollary 6.3.7 *A form in* S(a) *which is an eigenform for each* T_π *for* $\pi \nmid$ a *and each* U_π *and* W_π *for* $\pi \mid$ a, *is a newform.*

Corollary 6.3.8 *Suppose* $F \in$ S(a) *is an eigenform for all the* T_π *for* $\pi \nmid$ a *, and there exists* $\pi_0 \mid$ a *such that* F *is an eigenform for each* U_π *and* W_π *for* $\pi \mid$ a *except* $\pi = \pi_0$. *Then* F *is a member of some oldclass in* S(a) *defined by a newform* $G \in$ S(b) *where* b *divides* a *and* ab^{-1} *is a power of* (π_0).

Proof: Since F is an eigenform for T_π for each $\pi \nmid$ a it follows that F is a member of some oldclass, by the results stated at the end of §3.3. If this oldclass is defined by G, a newform in S(b), then the Proposition shows that only π_0 can divide ab^{-1}.

We now show how the twisting operators R_χ can, under suitable circumstances, produce newforms. For simplicity we restrict to the case where the conductor of χ is an odd prime ideal p; similar, but more complicated, results hold for other χ, which would have to be dealt with for each field in turn.

Let a be an ideal divisible by p to the exact power e, where $e \geq 2$ so that we can define R_χ on S(a). Let F be a newform in S(a) and set $F^* = F|R_\chi$. If F^* is not a newform, then as it is an eigenform for all the $T_{\pi'}$ for $\pi' \nmid$ a, and for $U_{\pi'}$ and $W_{\pi'}$ for $\pi' \mid$ a, except $\pi' = \pi$, by Corollary 6.3.8 we have

$$F^* = \sum_{r=0}^{f} x_r \left(G\left|\begin{pmatrix}\pi^r & 0\\ 0 & 1\end{pmatrix}\right. \right) \qquad (x_r \in \mathbb{C})$$

where G is a newform in S(b) and $b^{-1}a = p^f$ for some $f \geq 1$. If the Fourier coefficients of F* and G are $a(\alpha)$ and $b(\alpha)$ respectively, we thus have

(6.3.9) $$a(\alpha) = \sum_{r=0}^{f} x_r b(\alpha\pi^{-r}).$$

Taking $\alpha = 1$ we find that $a(1) = x_0 b(1) = x_0$ (since G is a newform and thus has first coefficient 1). If $\pi \mid \alpha$ then $a(\alpha) = 0$ since $\chi(\alpha) = 0$; if $\pi \nmid \alpha$ then (6.3.9) shows that

$$a(\alpha) = x_0 b(\alpha) = b(\alpha)$$

(where $x_0 = a(1) = 1$ since $a(1)$ is the first coefficient of F which is a newform). Hence we obtain the Fourier series of F* from that of G by deleting all terms corresponding to $\alpha \in O_K$ such that $\pi \mid \alpha$. Since $F = F^*|R_\chi$ (because $p^2 \mid a$, by Lemma 6.3.5) we thus have $F = G|R_\chi$.

This implies that F is a cusp form for $\Delta_0(c)$ where $c = bp^g$ and $g = \max(e-f,2) - (e-f)$, and so

$$(e-f) + \max(e-f,2) - (e-f) \geq e, \quad \text{or}$$

$$\max(e-f,2) \geq e.$$

The conditions $e \geq 2$ and $f \geq 1$ then give only the possibilities $e = 2$, $f = 1$ or 2.

Conversely, if G is a newform in $S(ap^{-1})$ or $S(ap^{-2})$ where p^2 divides a exactly, define $F = G|R_\chi$. This is certainly in S(a); it is an eigenform for all $U_{\pi'}$ and $W_{\pi'}$ for $\pi' \mid a$ and $(\pi) \neq (\pi')$. As for π itself: for each $\lambda \pmod{\pi}$ we can find λ' with $\chi(\lambda) = \chi(\lambda')$ such that $R_\pi^\lambda W_\pi R_\pi^{\lambda'} \in \Delta_0(ap^{-1})$ by Lemma 6.2.7 . Then

$$G|R_\pi^\lambda W_\pi = G|R_\pi^{-\lambda'}.$$

Summing over a set of invertible residues λ modulo π gives

$$(G|R_\chi)|W_\pi = \chi(-1)G|R_\chi$$

and so F is an eigenform for W_π also. Moreover $F|U_\pi = 0$ by Lemmas 6.3.4 and 6.3.5, so in fact F is an eigenform for <u>all</u> $W_{\pi'}$ and $U_{\pi'}$ for $\pi' \mid a$. By Corollary 6.3.7 it follows that F is in fact a <u>newform</u> in S(a).

We sum up the preceding discussion in the following.

Theorem 6.3.10 *Let χ be the quadratic character with conductor* $\mathfrak{p}$, *where* $\mathfrak{p}$ *is an odd prime ideal generated by an element* π *of* O_K. *Let* F *be a newform in* $S(\mathfrak{a})$ *where* $\mathfrak{p}^2$ *divides* $\mathfrak{a}$, *and set* $F^* = F|R_\chi$.

(i) *If* $\mathfrak{p}^3|\mathfrak{a}$ *then* F^* *is also a newform in* $S(\mathfrak{a})$, *possibly equal to* F.

(ii) *If* $\mathfrak{p}^2$ *divides* $\mathfrak{a}$ *exactly and* F^* *is not a newform in* $S(\mathfrak{a})$, *then there exists a newform* G *in* $S(\mathfrak{a}\mathfrak{p}^{-1})$ *or* $S(\mathfrak{a}\mathfrak{p}^{-2})$ *such that*

a) the Fourier series for F^* *is obtained from that of* G *by deleting the terms corresponding to* $\alpha \in O_K$ *such that* π *divides* α;

b) the eigenvalues of F^* *for all* $T_{\pi'}$ *(for* $\pi' \nmid \mathfrak{a}$*) and for* $W_{\pi'}$ *and* $U_{\pi'}$ *(for* $\pi' \mid \mathfrak{a}$ *and* $(\pi) \neq (\pi')$*) are the same as those of* G;

c) $F|W_\pi = \chi(\pi)F$.

Conversely, if G *is any newform in* $S(\mathfrak{a}\mathfrak{p}^{-2})$ *or* $S(\mathfrak{a}\mathfrak{p}^{-1})$ *where* $\mathfrak{p}^2$ *divides* $\mathfrak{a}$ *exactly, then* $F := G|R_\chi$ *is a newform in* $S(\mathfrak{a})$ *with eigenvalues derivable from those of* G *by Lemmas 6.2.6, 6.2.11 and 6.3.4 for* $(\pi) \neq (\pi')$, *and*

$$F|U_\pi = 0, \qquad F|W_\pi = \chi(\pi)F.$$

§6.4 Application: The Correspondence between V^+ and V^-

We end by applying the results of the previous sections of this Chapter to prove that, in the case of $\mathbb{Q}(\sqrt{-1})$, the connection between spaces of newforms in V^+ and V^- is indeed achieved by means of the twist R_2, as stated in §5.6. We also prove similar results for the other Euclidean fields.

Case $\mathbb{Q}(\sqrt{-1})$ Let $K = \mathbb{Q}(i)$, where $i = \sqrt{-1}$ as usual. Let $\chi : \mathbb{Z}[i] \to \{\pm 1\} \cup \{0\}$ be the character defined by

$$\chi(\alpha) = \begin{cases} 0 & \text{if } \alpha \equiv 0 \pmod{1+i}; \\ +1 & \text{if } \alpha \equiv 1 \pmod{2}; \\ -1 & \text{if } \alpha \equiv i \pmod{2}. \end{cases}$$

Then χ has conductor 2; write R_2 for the twisting operator R_χ. It is now clear that the Theorem of §5.6, which we restate here, is true.

Theorem 6.4.1 *Let* a *be an ideal of* $\mathbb{Z}[i]$ *such that* $(1+i)^4$ *divides* a. *Then the map* $R_2 : V(a) \to V(a)$ *satisfies*

(i) $R_2 J = -JR_2$;

(ii) $R_2 T_\pi = \left(\frac{\pi}{2}\right) T_\pi R_2$ *if* π *is prime,* $(\pi) \neq (1+i)$, *and* $(\pi) \nmid a$;

(iii) $R_2 W_\pi^e = \left(\frac{\pi}{2}\right)^e W_\pi^e R_2$ *if* π *is prime,* $(\pi) \neq (1+i)$, *and* $(\pi)^e \mid a$, and $\pi^{e+1} \nmid a$.

Proof: Part (i) follows from Proposition 6.2.11(iii) since $\chi(i) = -1$; similarly, parts (ii) and (iii) follow from Proposition 6.2.11(i) and (ii) since if $(\pi) \neq (1+i)$ then $\chi(\pi) = \left(\frac{\pi}{2}\right)$, the quadratic character modulo 2.

Case $\mathbb{Q}(\sqrt{-2})$ Set $K = \mathbb{Q}(\theta)$, where $\theta = \sqrt{-2}$; the unit group of $O_K = \mathbb{Z}[\sqrt{-2}]$ is generated by -1. To interchange V^+ and V^- we need to twist by R_χ for some character χ with $\chi(-1) = -1$, by 6.2.11(iii). There is no such character with conductor (2), but there are two possible characters modulo (4): they were called χ_2 and $\chi_1\chi_2$ in §6.1. Hence for $\mathbb{Q}(\theta)$ we have the following result.

Theorem 6.4.2 *Let* χ *be one of the two quadratic characters of* $\mathbb{Z}[\theta]$ *with conductor* $(\theta)^4$, *and* a *an ideal of* $\mathbb{Z}[\theta]$ *such that* $(\theta)^8$ *divides* a. *Then the map* $R_\chi : V(a) \to V(a)$ *satisfies*

(i) $R_\chi J = -JR_\chi$;

(ii) $R_\chi T_\pi = \chi(\pi) T_\pi R_\chi$ *if* $(\pi) \neq (\theta)$;

(iii) $R_\chi W_\pi = \chi(\pi)^e W_\pi R_\chi$ *if* $(\pi) \neq (\theta)$, and π divides a to the exact power e.

As a corollary of (i), we see that R_χ interchanges $V^+(a)$ and $V^-(a)$ for an ideal a such that $(\theta)^8$ divides a. To illustrate this, we refer to Tables 5.2.2 and 5.2.3. There is a newform in $V^+(\theta^5)$, with a corresponding newform in $V^-(\theta^8)$; also there is a conjugate pair of newforms in $V^-(\theta^7)$, whose twists occur as a conjugate pair of newforms in $V^+(\theta^8)$.

Case $\mathbb{Q}(\sqrt{-3})$ Recall that $\rho = \frac{1}{2}(1+\sqrt{-3})$. We have the following result.

Theorem 6.4.3 *Let* χ *be one of the two quadratic characters of* $\mathbb{Z}[\rho]$

with conductor $(2)^2$ *such that* $\chi(\rho) = -1$, *and let* $\mathfrak{a}$ *be an ideal of* $Z[\rho]$ *such that* $(2)^4$ *divides* $\mathfrak{a}$; *then the map* $R_\chi : V(\mathfrak{a}) \to V(\mathfrak{a})$ *satisfies*

(i) $R_\chi J = - JR_\chi$;

(ii) $R_\chi T_\pi = \chi(\pi) T_\pi R_\chi$ *if* $(\pi) \neq (2)$;

(iii) $R_\chi W_\pi = \chi(\pi)^e W_\pi R_\chi$ *if* $(\pi) \neq (2)$, and π divides $\mathfrak{a}$ to the exact power e.

There are no illustrations of such a twist in §5.3, because the only ideal $\mathfrak{a}$ in the range of the tables divisible by $(2)^4$ is $(2)^4$ itself, and $\dim V((2)^4) = 0$.

Case $Q(\sqrt{-7})$ Write $\alpha = \frac{1}{2}(1+\sqrt{-7})$ and let λ be one of the primes dividing 2 in $Z[\alpha]$; so $\lambda = \alpha$ or $\lambda = \bar{\alpha}$. There is a quadratic character χ with conductor $(\alpha)^2$ such that $\chi(-1) = -1$, and similarly the conjugate character $\bar{\chi}$ has conductor $(\bar{\alpha})^2$. So we have the following result.

Theorem 6.4.4 *Let* ψ *be one of the characters* $\chi, \bar{\chi}$ *with conductor* $(\lambda)^2$, *say. Let* $\mathfrak{a}$ *be an ideal of* $Z[\alpha]$ *such that* $(\lambda)^4$ *divides* $\mathfrak{a}$. *Then the map* $R_\psi : V(\mathfrak{a}) \to V(\mathfrak{a})$ *satisfies*

(i) $R_\psi J = - JR_\psi$;

(ii) $R_\psi T_\pi = \psi(\pi) T_\pi R_\psi$ *if* $(\pi) \neq (\lambda)$;

(iii) $R_\psi W_\pi = \psi(\pi)^e W_\pi R_\psi$ *if* $(\pi) \neq (\lambda)$, and π divides $\mathfrak{a}$ to the exact power e.

To illustrate this, we refer to Tables 5.4.2 and 5.4.3. There is a newform in $V^+((7+3\alpha))$ whose twist is a newform in $V^-((13-7\alpha))$: here, $(7+3\alpha) = (1+2\alpha)(\bar{\alpha})^3$ and $(13-7\alpha) = (1+2\alpha)(\bar{\alpha})^4$; in Table 5.4.3, the newform conjugate to the latter is given, in $V^-((6+7\alpha))$. Also, there is a newform in $V^+((2-7\alpha))$ with a corresponding newform in $V^-((14-5\alpha))$, whose conjugate in $V^-((9+5\alpha))$ is given in Table 5.4.3. Here, $(2-7\alpha) = (\alpha)^3(1+2\alpha)$ and $(14-5\alpha) = (\alpha)^4(1+2\alpha)$.

Case $Q(\sqrt{-11})$ Set $\alpha = \frac{1}{2}(1+\sqrt{-11})$. Now (2) is prime; there are no quadratic characters with conductor (2), but there are three with conductor $(2)^2$, of which two satisfy $\chi(-1) = -1$. Hence we have the following.

Theorem 6.4.5 *Let χ be one of the quadratic characters of $\mathbb{Z}[\alpha]$ with conductor $(2)^2$, satisfying $\chi(-1) = -1$. Let $\mathfrak{a}$ be an ideal of $\mathbb{Z}[\alpha]$ with* $(2)^4 \mid \mathfrak{a}$. *Then* $R_\chi : V(\mathfrak{a}) \to V(\mathfrak{a})$ *satisfies*

(i) $JR_\chi = - R_\chi J$;

(ii) $T_\pi R_\chi = \chi(\pi) R_\chi T_\pi$ *if* $(\pi) \neq (2)$;

(iii) $W_\pi R_\chi = \chi(\pi)^e R_\chi W_\pi$ *if* $(\pi) \neq (2)$, and π divides $\mathfrak{a}$ to the exact power e.

There are no examples of these twists in Tables 5.5.2 and 5.5.3, since none of the levels $\mathfrak{a}$ covered by those tables is divisible by $(2)^4$.

We finish by giving one last example from $\mathbb{Q}(i)$ to illustrate the correspondence between V^+ and V^-, the correspondence between V^+ and elliptic curves, and to show how twisting a newform corresponds to twisting the corresponding curve.

At level (10) for $\mathbb{Q}(i)$, there is a newform in $V^+((10))$, and a corresponding elliptic curve $y^2 = x^3 + x^2 - x$ (number 6 in Table 5.1.4). Applying the (2i-1)-twist, we obtain a newform at level (10)(2i-1) = (20i-10), and because i is not a square modulo (2i-1), this newform is in $V^-((20i-10))$. There is no corresponding curve. If we then apply R_2, in order to obtain a newform in V^+, to the latter, we find, as expected, a newform at level $(20i-10) \cap (1+i)^4 = (40i-20)$, in $V^+((40i-20))$. Moreover, if we twist the original curve by 2i-1 we obtain the curve

$$y^2 = x^3 + (2i-1)x^2 - (2i-1)^2 x$$

which has conductor (40i-20) and corresponds to the latter newform.

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